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Mathematics: A Second Level Course
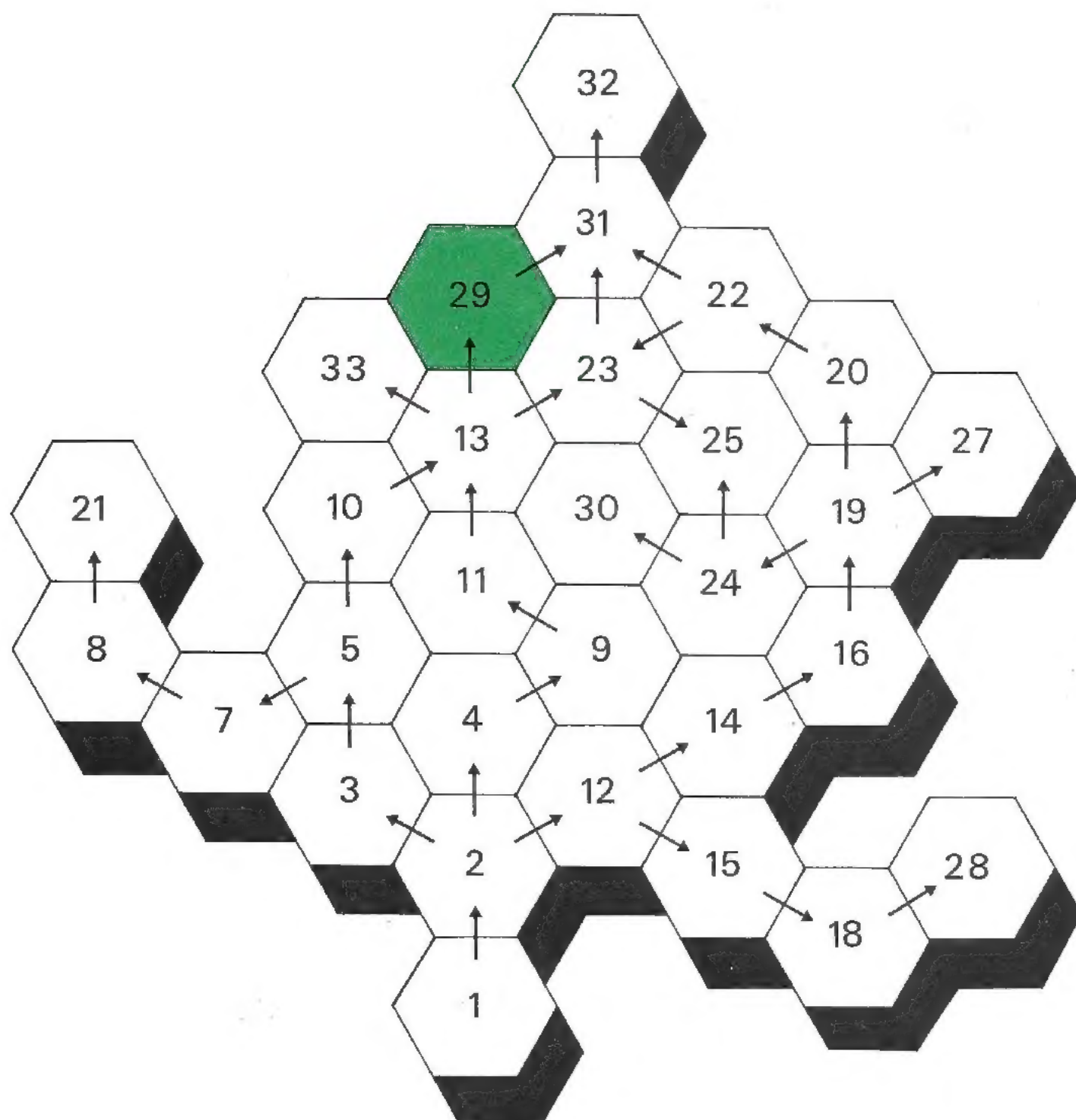
Linear Mathematics Unit 29

Laplace Transforms

Mathematics: A Second Level Course

Linear Mathematics   Unit 29

LAPLACE TRANSFORMS

Prepared by the Linear Mathematics Course Team

This text forms part of a series of units that make up the correspondence
element of an Open University Second Level course. The complete list of
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Contents

Set Books

D. L. Kreider, R. G. Kuller, D. R. Ostberg and F. W. Perkins, *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E. D. Nering, *Linear Algebra and Matrix Theory* (John Wiley, 1970).

It is essential to have these books; the course is based on them and will not make sense without them.

Conventions

Before working through this correspondence text make sure you have read *A Guide to the Linear Mathematics Course*. Of the typographical conventions given in the Guide the following are the most important.

The set books are referred to as:

       K for *An Introduction to Linear Analysis*
       N for *Linear Algebra and Matrix Theory*

All starred items in the summaries are examinable.

References to the Open University Mathematics Foundation Course Units (The Open University Press, 1971) take the form *Unit M100 3, Operations and Morphisms.*

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29.0 INTRODUCTION

This unit describes a method of solution for linear differential equations which is specifically designed to solve initial-value problems, and which is often more powerful for such problems than the more general methods we studied in *Unit 9, Differential Equations II, Unit 11, Differential Equations III*, and *Unit 13, Systems of Differential Equations*. In this unit we shall be more concerned with the rudiments of the theory than with actually solving difficult problems.

The basic idea of the method is this. Suppose f is a bounded function in $C[0, \infty)$. Its *Laplace transform** is the function ϕ defined by

$$\phi(s) = \int_0^\infty e^{-st} f(t) \, dt,$$

where $\int_0^\infty$ means $\lim_{t_0 \text{ large}} \int_0^{t_0}$, for all real values of s for which this limit exists. The mapping

$$\mathfrak{L}: f \longmapsto \phi$$

with domain a suitable subspace of $C[0, \infty)$ is a linear transformation, that is, a morphism with respect to linear combinations. The important thing here, however, is that under suitable conditions this mapping $\mathfrak{L}$ is also a morphism with respect to the unary operation of differentiation. It turns out for suitable functions f that the Laplace transform of f', the derived function of f, is[†]

$$\mathfrak{L}[f']: s \longmapsto \int_0^\infty e^{-st} f'(t) \, dt = s\mathfrak{L}[f](s) - f(0)$$

If in addition $f(0) = 0$,

$$\mathfrak{L}[f']: s \longmapsto s \int_0^\infty e^{-st} f(t) \, dt = s\phi(s),$$

where $\phi = \mathfrak{L}[f]$.

Thus for a class of functions satisfying certain conditions the following diagram commutes:

where $s\phi$ means the function

$$s \longmapsto s\phi(s)$$

Because of this morphism, the Laplace transform is of great value for solving differential equations, since it maps the operation of differentiation to the purely algebraic operation of multiplying by the function $s \longmapsto s$. If f satisfies a differential equation, the Laplace transform will map this to an algebraic equation for ϕ, and this algebraic equation is easier to

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* Marquis de Pierre Simon Laplace (1749–1827) was not only a great French mathematician and philosopher but also played a prominent part in French politics. He was, at one time, the Minister of the Interior under Napoleon, a post from which he was dismissed after a mere six weeks for "bringing the spirit of infinitesimals into administration" and elevated to a place in Senate. Although he is most celebrated for his work on gravitation it was in his analytic study of probability theory in the early 19th century that he developed the theory of Laplace transforms.

† We write $[f']$ rather than (f') to fit in with the notation of K.

solve than the original differential equation. For example, if f satisfies

$$f' + f = h$$

and

$$f(0) = 0$$

where h is some given function, then ϕ the Laplace transform of f will satisfy

$$s\phi(s) + \phi(s) = \mathcal{L}[h](s).$$

This is an algebraic equation for ϕ, whose solution is

$$\phi(s) = (s + 1)^{-1}\mathcal{L}[h](s).$$

To arrive at the solution of the original differential equation, therefore, we have only to calculate the *inverse Laplace transform* of ϕ, i.e. the function whose Laplace transform is $s \longmapsto (s + 1)^{-1}\mathcal{L}[h](s)$. There are well-established techniques for finding inverse Laplace transforms, and we shall study some of them in this unit.

29.1 THE DEFINITION OF THE MAPPING £

29.1.0 Introduction

We have seen above that the Laplace transform of a function f is given by the formula

$$\pounds[f]: s \longmapsto \int_0^\infty e^{-st} f(t)\, dt$$

but in order to define £ properly we should define its domain and codomain as well as its formula. In this section we look at these three elements of the definition.

29.1.1 Piecewise Continuous Functions

We want to be able to apply Laplace transform methods to as wide a class of functions as possible. As in *Unit 22, Fourier Series*, it turns out that we need not require f to be continuous; it is enough for it to be piecewise continuous. Indeed, one of the advantages of Laplace transform methods over the conventional method is that it requires no modification for piecewise continuous functions. To review the definition of piecewise continuity,

READ Section 5-1, pages K177–8.

Notes

(i) *lines 17-20, page* K178   The restriction $t \in [0, \infty)$ describes the domains of the functions to which we apply the Laplace transform. All functions in the domain of £ have $[0, \infty)$ as *their* domain. In the notation of *Unit 22*, the domain of £ is a subset of $\mathcal{PC}[0, \infty)$.

(ii) *lines 21-23, page* K178 "piecewise continuous on *every* finite interval". For example, the square-wave function shown on page K177, which is periodic with period $2a$, has an infinite number of discontinuities altogether; but it only has a finite number on any finite interval (of the t-axis) and therefore is piecewise continuous on $[0, \infty)$.

Example

Consider the following electrical circuit. It contains an inductance L, a
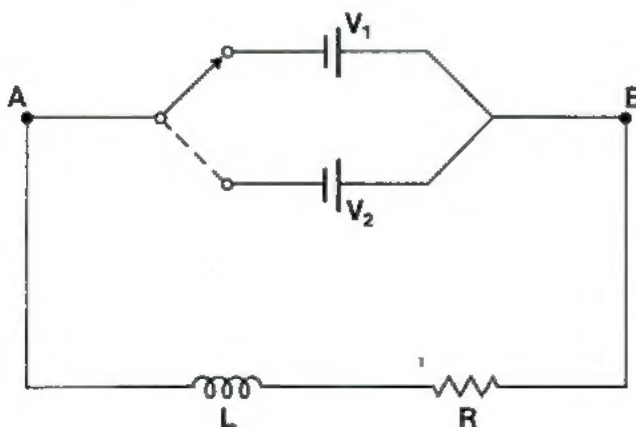
resistance R, and a switch that can connect either of two batteries of voltage V_1 and V_2. The function

$$V: t \longmapsto V(t) \qquad (t \in R)$$

where $V(t) = V_i$ if the switch is connected to the ith battery at time t, describes the voltage between points A and B. If the switch is switched to

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and fro, then V will be a piecewise continuous function. The differential equation describing the current i in the circuit is

$$Li'(t) + Ri(t) = V(t) \qquad t \in R,$$

and to solve this equation using the Laplace transform we would have to calculate

$$\int_0^\infty e^{-st} V(t) \, dt$$

which involves integrating a piecewise continuous function.

Exercises

1. Exercise 1, page K178.

2. Exercise 2, page K178, omitting part (b).

$$\left(Hint: \quad \lim_{t \to 0} \frac{\sin t}{t} = 1 \right)$$

Solutions

1. See page K739.

2. (a) Piecewise continuous only if $n = 1$. This is because

   $$\lim_{t \to 0} \frac{\sin t}{t} = 1,$$

   but

   $$\lim_{t \to 0} \frac{\sin t}{t^n}$$

   does not exist for $n > 1$; so the limit of $f(0 + h)$ as h tends to zero through positive values (see page K177, line -6) exists only if $n = 1$.

   (c) Piecewise continuous on $[0, \infty)$ (see page K178, lines 21 to 23). There are an infinite number of discontinuities, but any finite interval $[a, b]$ only contains a finite number of them (this number is less than $b - a + 1$).

   (d) Not piecewise continuous on $[0, \infty)$, because there are an infinite number of discontinuities in the interval $[0, 1]$. (However, for any $\varepsilon > 0$, the function is piecewise continuous on $[\varepsilon, \infty)$.)

   (e) Piecewise continuous on $[0, \infty)$, for the same reason as in (c).

29.1.2 A New Type of Integral

One of the new features of the integral defining the Laplace transform is its upper endpoint, which we have defined by

$$\int_0^\infty f \text{ means } \lim_{t_0 \text{ large}} \int_0^{t_0} f.$$

provided the limit exists. An equivalent notation is

$$\lim_{t_0 \to \infty} \int_0^{t_0} f.$$

If the limit does exist the integral $\int_0^\infty f$ is said to *converge*; if not, it is said to *diverge*.

For example,

$$\int_1^\infty e^{-t}\,dt = \lim_{t_0 \to \infty} \int_1^{t_0} e^{-t}\,dt$$

$$= \lim_{t_0 \to \infty} [-e^{-t}]_1^{t_0}$$

$$= \lim_{t_0 \to \infty} (-e^{-t_0} + e^{-1})$$

$$= e^{-1}.$$

Thus $\int_1^\infty e^{-t}\,dt$ converges to e^{-1}.

On the other hand,

$$\int_3^\infty t\,dt = \lim_{t_0 \to \infty} \left[\frac{t^2}{2}\right]_3^{t_0}$$

$$= \lim_{t_0 \to \infty} \tfrac{1}{2}(t_0^2 - 9)$$

which does not exist. Thus $\int_3^\infty t\,dt$ diverges.

Example

We calculate the Laplace transform of

$$f: t \longmapsto 1 \qquad (t \in R).$$

Its transform $\mathcal{L}[f]$ maps s to

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} \times 1\,dt$$

$$= \lim_{t_0 \to \infty} \int_0^{t_0} e^{-st}\,dt$$

$$= \lim_{t_0 \to \infty} \left[-\frac{1}{s} e^{-st}\right]_0^{t_0}$$

$$= \lim_{t_0 \to \infty} \left(-\frac{1}{s} e^{-st_0}\right) + \frac{1}{s}.$$

Provided $s > 0$, we can make e^{-st_0} as small as we please by making t_0 large enough, and so

$$\lim_{t_0 \to \infty} \left(-\frac{1}{s} e^{-st_0}\right) = 0.$$

That is

$$\mathcal{L}[f](s) = \frac{1}{s}$$

if $s > 0$. It is typical of Laplace transforms that the integral converges only for s larger than some number, here 0.

READ Section 5-2, page K179, as far as Equation (5-2) on page K180.

Note

line -4, *page* K179 $\mathcal{L}[\cos at]$ which stands for

$$\mathcal{L}[t \longmapsto \cos at]$$

is a function.

Exercises

I. Show that only one of the following integrals is convergent and evaluate it.

(a) $\int_2^\infty \frac{1}{t^2}\,dt$

(b) $\displaystyle\int_0^\infty \sin t\, dt$

2. Find the Laplace transform (for $s > 0$) of

$$f : t \longmapsto t \qquad (t \in R)$$

(*Hint*: $\lim\limits_{t_0 \to \infty} t_0 e^{-st_0} = 0$ if $s > 0$. This will be proved in sub-section 29.1.3.)

3. Find $\mathcal{L}\,[\sin at](s)$.

Solutions

1. Integral (a) is convergent; (b) is divergent.

(a) $\displaystyle\int_2^\infty \frac{1}{t^2}\, dt = \lim_{t_0 \to \infty} \int_2^\infty \frac{1}{t^2}\, dt$

$$= \lim_{t_0 \to \infty} \left[-\frac{1}{t} \right]_2^{t_0}$$

$$= \lim_{t_0 \to \infty} \left(-\frac{1}{t_0} \right) + \frac{1}{2}$$

$$= \frac{1}{2}.$$

(b) $\displaystyle\int_0^\infty \sin t\, dt = \lim_{t_0 \to \infty} \int_0^{t_0} \sin t\, dt$

$$= \lim_{t_0 \to \infty} (1 - \cos t_0).$$

Since $1 - \cos t_0$ oscillates perpetually between 0 and 2, the integral diverges.

2. $\displaystyle \mathcal{L}[f]s = \int_0^\infty e^{-st} t\, dt$

$$= \lim_{t_0 \to \infty} \left[e^{-st} \left(-\frac{t}{s} - \frac{1}{s^2} \right) \right]_0^{t_0}, \text{ by Section III.5.2 of TI}$$

$$= \lim_{t_0 \to \infty} e^{-st_0} \left(-\frac{t_0}{s} - \frac{1}{s^2} \right) + \frac{1}{s^2}$$

$$= \frac{1}{s^2} \text{ if } s > 0.$$

3. $\displaystyle \mathcal{L}\,[\sin at](s) = \int_0^\infty e^{-st} \sin at\, dt$

$$= \lim_{t_0 \to \infty} \int_0^{t_0} e^{-st} \sin at\, dt$$

$$= \lim_{t_0 \to \infty} \left[\frac{e^{-st}}{s^2 + a^2} (-a \cos at - s \sin at) \right]_0^{t_0},$$

by Section III.5.2 of TI,

$$= \lim_{t_0 \to \infty} \left(\frac{e^{-st_0}}{s^2 + a^2} (-a \cos at_0 - s \sin at_0) \right)$$

$$+ \frac{a}{s^2 + a^2}$$

$$= \frac{a}{s^2 + a^2}.$$

29.1.3 Functions of Exponential Order

In order to define $\mathcal{L}$ properly, we must specify its domain. We have already indicated that this domain will be a set of functions that are piecewise continuous on $[0, \infty)$. We cannot, however, allow all such functions to belong to the domain of $\mathcal{L}$, because we want the integral

$$\int_0^\infty e^{-st}f(t)\,dt$$

to converge for at least some value s, and there are functions (for example $t \longmapsto \exp(t^2)$) which increase so rapidly with t that the integral does not converge for any s. You will see in the next reading passage a simple condition of f which is sufficient (though not necessary) to *ensure* that the integral converges.

READ from line 5 of page **K**180 *to the end of Section 5-2 on page* **K**182.

Notes

(i)   *line 1, page* **K**181   L'Hôpital's rule does not form part of this course. It is a set of rules for finding limits such as

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}, \qquad \lim_{t \to \infty} \frac{f(x)}{g(x)}.$$

For our purposes the following will serve equally well.

Theorem

If n is a non-negative integer and a is a positive real number, then

$$\lim_{t \to \infty} t^n e^{-at} = 0.$$

Proof

The Taylor series for the exponential function (*Unit M100 14, Sequences and Limits II*) is

$$e^{at} = 1 + at + \cdots + \frac{(at)^n}{n!} + \frac{(at)^{n+1}}{(n+1)!} + \cdots \quad (t \in R)$$

If $at > 0$, all the terms in the series are positive, so that

$$e^{at} > \frac{(at)^{n+1}}{(n+1)!}.$$

It follows that

$$t^n e^{-at} = \frac{t^n}{e^{at}} < \frac{t^n(n+1)!}{(at)^{n+1}}$$

$$= \frac{(n+1)!}{a^{n+1}} \times \frac{1}{t}$$

which can be made as close to zero as we please by making t large enough. We conclude that as t becomes large, $t^n e^{-at}$ must approach 0.

(ii)   *lines -12 to -9 page* **K**181   The following diagram may help to make the comparison theorem plausible. The broken line shows the graph of f where it differs from that of $|f|$.
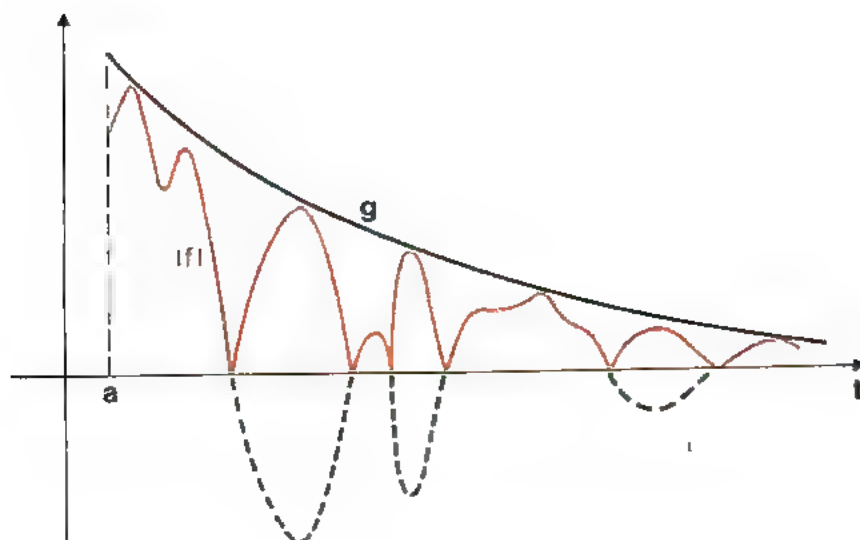
(iii) *footnote, page* K181   The comparison test for series is Theorem 1-6 on page K644. It was used in *Unit 20, Euclidean Spaces II*, sub-section 20.1.4.

(iv) *lines 2-3, page* K182   "greatest lower bound ... all $s > \alpha$". You may prefer to think of s_0 as the greatest lower bound of the set of real numbers s such that

$$\int_0^\infty e^{-st} f(t)\, dt$$

converges. For example, we saw in the previous sub-section (Equation 5-2) that if

$$f(t) = \cos at$$

then the integral converges for $s > 0$ but not for $s \le 0$. In this case the set for which the integral converges is $\mathcal{R}^+$. Every negative number is a lower bound on this set, and so is zero; so $s_0 = 0$ for $t \longmapsto \cos at$.

A further discussion of greatest lower bounds can be found on page K637 and in *Unit M100 19, Relations*.

(v) *line 7 page* K182   "s_0 may be $-\infty$". This means that in such cases the domain of $\mathcal{L}[f]$ is $(-\infty, \infty)$: the integral converges for all real s.

Exercises

Exercises 1, 2, 4, 5 on page K182.

Solutions

1 on page K182

The Laplace transform (calculated in Exercise 2 of the preceding sub-section) is

$$s \longmapsto \frac{1}{s^2} \qquad (s > 0)$$

If $s \le 0$ the integral diverges, and so the abscissa of convergence is 0.

2 on page K182

$$\mathcal{L}[e^{at}](s) = \lim_{t_0 \to \infty} \int_0^{t_0} e^{-st} e^{at}\, dt$$

$$= \lim_{t_0 \to \infty} \left[\frac{e^{(a-s)t}}{a-s}\right]_0^{t_0} \quad \text{if } s \ne a$$

$$= \lim_{t_0 \to \infty} \left(\frac{e^{(a-s)t_0}}{a-s}\right) - \frac{1}{a-s}$$

$$= \frac{1}{s-a} \text{ if } s > a.$$

Since the integral converges if $s > a$ and diverges for $s \leqslant a$ the abscissa of convergence is a.

4 on page K182

By the example in sub-section 29.1.2

$$\mathfrak{L}[1](s) = \frac{1}{s} \text{ if } s > 0.$$

and the integral diverges if $s \leqslant 0$. The abscissa of convergence is 0.

5 on page K182

In Exercise 3 of sub-section 29.1.2 we saw that

$$\mathfrak{L}\,[\sin at](s) = \frac{a}{s^2 + a^2} \text{ if } s > 0.$$

The integral diverges if $s < 0$ and so the abscissa of convergence is 0.

29.1.4   The Codomain of $\mathfrak{L}$

If $\mathfrak{L}$ is to be considered as a linear transformation, then its codomain must be a vector space. In this course we have dealt with many examples of vector spaces of functions. In most cases hitherto, every function in the vector space had the same domain. For example, every function in $C[0, 1]$ has $[0, 1]$ as its domain, and every function in $C^\infty(R)$ has R as its domain.

The functions we obtain by applying the Laplace transform, however, can not all have the same domain. For example, we saw in the preceding sub-section that

$$\int_0^\infty e^{-st} e^{at}\, dt = \frac{1}{s - a} \qquad (s > a)$$

but that the integral diverges if $s \leqslant a$; so the Laplace transform of

$$t \longmapsto e^{at} \qquad (t \in [0, \infty))$$

is

$$s \longmapsto \frac{1}{s - a} \qquad (s \in (a, \infty)),$$

and the domain of the image function depends on a, that is on the function whose Laplace transform we take. This leads to some technical difficulties in defining a suitable vector space structure in the codomain of $\mathfrak{L}$; for up to now we have usually defined addition of functions when the functions added have the same domain.

For the domain of $\mathfrak{L}$ we take $\mathscr{E}$, the set of all piecewise continuous functions of exponential order. With the usual definitions of addition and scalar multiplication, $\mathscr{E}$ is a real vector space. For the codomain of $\mathfrak{L}$ we consider the set $\mathscr{F}$ of all real-valued functions with intervals of the form (s_0, ∞), $[s_0, \infty)$ or $(-\infty, \infty)$ as domain. To accommodate the fact that the members of $\mathscr{F}$ do not all have the same domain, we define addition in $\mathscr{F}$ as follows: if $f, g \in \mathscr{F}$, $f + g$ is the function whose domain is the *intersection* S of the domains of f and g, such that

$$(f + g)(s) = f(s) + g(s), \qquad s \in S.$$

Having done this, we can see that $\mathcal{F}$ is not a vector space since there is not a unique zero element. For example if

$$f: x \longmapsto 1 \qquad (x \in [1, \infty))$$

$$g: x \longmapsto 0 \qquad (x \in [1, \infty))$$

$$h: x \longmapsto 0 \qquad (x \in [0, \infty))$$

then

$$f + g = f$$

and

$$f + h = f.$$

Since $\mathcal{F}$ is not a vector space, $\mathcal{L}: \mathcal{E} \longrightarrow \mathcal{F}$ cannot be a linear transformation. Thus there exist f and g such that

$$\mathcal{L}[f + g] \neq \mathcal{L}[f] + \mathcal{L}[g].$$

For example, let

$$f(t) = \cos at$$

$$g(t) = -\cos at$$

By Equation (5-2), page K180

$$\mathcal{L}[f]: s \longmapsto \frac{s}{s^2 + a^2} \qquad (s \in (0, \infty))$$

$$\mathcal{L}[g]: s \longmapsto \frac{-s}{s^2 + a^2} \qquad (s \in (0, \infty))$$

so that

$$\mathcal{L}[f] + \mathcal{L}[g]: s \longmapsto 0 \qquad (s \in (0, \infty))$$

but

$$\mathcal{L}[f + g](s) = \int_0^\infty e^{-st} \times 0 \, dt = 0 \text{ for all real } s$$

so that

$$\mathcal{L}[f + g]: s \longmapsto 0 \qquad (s \in (-\infty, \infty)).$$

Thus $\mathcal{L}[f] + \mathcal{L}[g]$ and $\mathcal{L}[f + g]$ have different domains and are therefore different functions.

READ from line 9, page K184 " From this ..." to line 20, page K184.

Notes

(i) *lines 10 to 11 page* K184 "$\mathcal{L}[f + g]$ and $\mathcal{L}[f] + \mathcal{L}[g]$ are identical for those values of s where both these functions are defined" means "$\mathcal{L}[f + g]$ and $\mathcal{L}[f] + \mathcal{L}[g]$ map s to the same number for each s in the domain of both these functions."

(ii) *lines 14, 15 page* K184 "*identical* whenever they coincide on an interval of the form (a, ∞)." This is a similar device to the one used in *Unit 22, Fourier Series*, to define the Euclidean spaces $\mathcal{SC}[a, b]$. We define a relation on the set $\mathcal{F}$ by the rule

$f \sim g$ if there exists a real number a such that $f(s) = g(s)$ for all $s > a$.

It is not hard to show that this relation is an equivalence relation, and so we may say that f and g are equivalent if $f \sim g$, or as K says, the functions f and g are *identical* or *identified*. Note that different pairs of equivalent functions may coincide for different choices of a. For example, the functions

$$s \longmapsto 0 \qquad (s \in (0, \infty))$$

and

$$s \longmapsto 0 \qquad (s \in (-\infty, \infty))$$

* Note that we have omitted the early part of Section 5.3 of K in favour of the paragraph in this text preceding the READ instruction.

are equivalent (identical) since they both map all positive numbers to 0. Likewise the functions $s \longmapsto s$ and $s \longmapsto |s|$ both with domain $(-\infty, \infty)$, are equivalent because every *positive* number s is mapped by both functions to the same image. Note carefully that we do not (indeed cannot) require that the images of the functions coincide on the *entire* intersection of their domains.
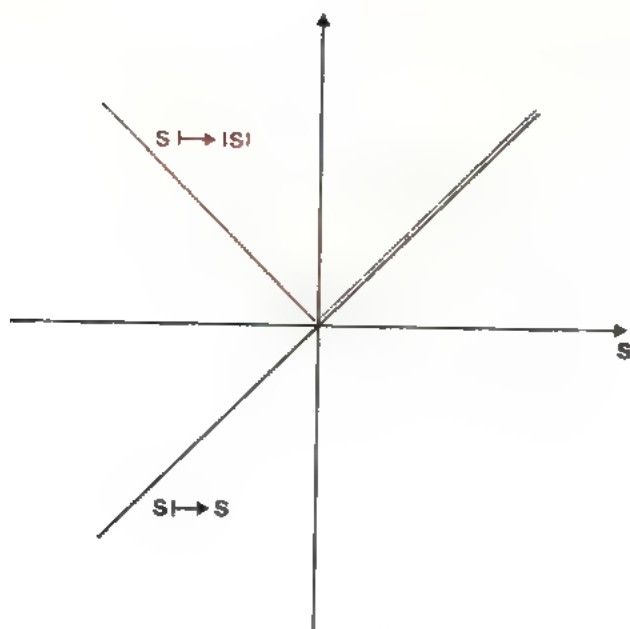
(iii) *line·17, page* K184 "enforcing this identification". This means that from now on K will write $f = g$ (an abuse of notation) rather than $f \sim g$ to mean that the functions f and g are equivalent.

(iv) *footnote, page* K184 This amounts to saying that K will write $\mathcal{F}$ to stand either for the space $\mathcal{F}^*$ of equivalence classes or for $\mathcal{F}$ itself, according to context. In this text, however, we shall use $\mathcal{F}^*$ for the space of equivalence classes.

Exercises

1. Determine the functions

$$\mathcal{L}[e^t], \mathcal{L}[e^{-t}], \mathcal{L}[e^t] + \mathcal{L}[e^{-t}].$$

2. Which two of the following functions are equivalent under the relation $\sim$?

$$f: s \longmapsto \begin{cases} \dfrac{1}{s-3} & \text{if } s < 2 \\[3mm] \dfrac{1}{s} & \text{if } s \geqslant 2 \end{cases} \qquad (s \in (-\infty, \infty))$$

$$g: s \longmapsto \begin{cases} \dfrac{1}{s} & \text{if } s < 3 \\[3mm] \dfrac{1}{s-3} & \text{if } s \geqslant 3 \end{cases} \qquad (s \in (0, \infty))$$

$$h: s \longmapsto \begin{cases} 1 & \text{if } s < 4 \\[3mm] \dfrac{1}{s-3} & \text{if } s \geqslant 4 \end{cases} \qquad (s \in [1, \infty)).$$

3. Prove that $\sim$ is an equivalence relation.

Solutions

1. From the exercises to subsection 29.1.3, we have

$$\mathcal{L}[e^t]: s \longmapsto \frac{1}{s-1} \qquad (s \in (1, \infty))$$

$$\mathcal{L}[e^{-t}]: s \longmapsto \frac{1}{s+1} \qquad (s \in (-1, \infty))$$

and so

$$\mathcal{L}[e^t] + \mathcal{L}[e^{-t}]: s \longmapsto \frac{1}{s-1} + \frac{1}{s+1} \qquad (s \in (1, \infty))$$

since

$$(1, \infty) \cap (-1, \infty) = (1, \infty).$$

2. We have

$$g(s) = h(s) = \frac{1}{s-3} \text{ for all } s > 4 \text{ so that } g \sim h.$$

3. *Reflexive*

   "$f \sim f$" follows directly from the definition.

   Symmetric

   "If $f \sim g$, then $g \sim f$" also follows directly from the definition.

   Transitive

   Now, if $f \sim g$, then for some $c \in R$

   $$f(x) = g(x) \qquad x \in (c, \infty)$$

   Similarly, if $g \sim h$, then for some $d \in R$

   $$g(x) = h(x) \qquad x \in (d, \infty)$$

   Then if b is the larger of c and d we must have

   $$f(x) = g(x) = h(x) \qquad x \in (b, \infty)$$

   and so $f \sim h$.

29.1.5 Can We Invert $\mathcal{L}$?

We have mentioned (in the Introduction) that in order to use the Laplace transform to solve differential equations we must be able to invert it; that is, given a function ϕ in $\mathcal{F}^*$ we want a unique function f in $\mathcal{E}$ such that $\mathcal{L}^*[f] = \phi$. Whether this can always be done depends on whether $\mathcal{L}$ is an isomorphism, i.e. whether $\mathcal{L}$ is one-to-one and onto. These two points are discussed in the next reading passage.

READ from line 21 of page K184 to the end of Section 5-3.

Notes

(i) *line −6, page* K184 "differ *only* at their points of discontinuity". Two functions f and g with the property mentioned are equivalent under the equivalence relation we used to define $\mathcal{FC}[a, b]$ in *Unit 22, Fourier Series*, (page K332). We write this as fpg. For example, the functions

$$f: t \longmapsto 0 \qquad (t \in [0, \infty))$$

and

$$g: t \longmapsto \begin{cases} 1 \text{ if } t \in Z \\ 0 \text{ if not} \end{cases} \qquad (t \in [0, \infty))$$

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are equivalent under this equivalence relation, so that $f \rho g$. This is an equivalence relation on $\mathcal{E}$, and has nothing to do with the equivalence relation on $\mathcal{F}$ defined in sub-section 29.1.4. We will write $\mathcal{E}^*$ for the set of equivalence classes under ρ. (Thus the mapping under discussion is really

$$\mathcal{L}^* : \mathcal{E}^* \longrightarrow \mathcal{F}^*.)$$

(ii) *Lerch's theorem, page* K185   This can be stated in terms of the two equivalence relations we have discussed:

> if $f, g \in \mathcal{E}$ and $\mathcal{L}[f] \sim \mathcal{L}[g]$, then $f \rho g$.

(iii) *Last paragraph of the discussion on Theorem 5-2, page* K185   The conclusion is that $\mathcal{L}^*$ is one-to-one but not onto; it is therefore not an isomorphism of $\mathcal{E}^*$ to $\mathcal{F}^*$. If $\mathcal{L}^{-1}[\phi]$ exists, it is unique (i.e. a unique equivalence class in $\mathcal{E}^*$,) but it does not exist for every ϕ in $\mathcal{F}^*$. Thus $\mathcal{L}^{-1}$ is *not* a function on $\mathcal{F}^*$.

Example

We have seen in Exercise 1 of sub-section 29.1.4 that

$$\mathcal{L}[e^t + e^{-t}] : s \longmapsto \frac{1}{s-1} + \frac{1}{s+1} = \frac{2s}{s^2-1} \qquad (s \in (1, \infty)).$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{2s}{s^2-1}\right] : t \longmapsto e^t + e^{-t}$$

or, more precisely, that

$$\mathcal{L}^{-1}\left[\frac{2s}{s^2-1}\right] \text{ is the equivalence class}$$

$$\{f : f \in \mathcal{E} \text{ and } f \rho(t \longmapsto e^t + e^{-t})\}.$$

Exercise

Using the solutions to the exercises of sub-section 29.1.3, find the inverse images:

(i) $\mathcal{L}^{-1}[g]$, where

$$g(s) = \frac{1}{s-a} + \frac{a}{s^2+a^2} \qquad (s \in (a, \infty)),$$

and where a is a real positive number;

(ii) $\mathcal{L}^{-1}[h]$, where

$$h(s) = \frac{1}{s-a} \quad (a < s < b) \Bigg\} \qquad (s \in (a, \infty))$$
$$= 0 \quad (s \geqslant b)$$

and where a and b are real numbers with $a < b$. (Don't jump to conclusions. Read the question carefully before trying it.)

Solution

(i)   We know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad (s > a)$$

and

$$\mathcal{L}[\sin at] = \frac{a}{s^2+a^2} \qquad (s > 0).$$

Thus

$$\mathcal{L}[e^{at} + \sin at] = \frac{1}{s-a} + \frac{a}{s^2+a^2} \qquad (s > a)$$

and

$$\mathcal{L}^{-1}\left[\frac{1}{s-a} + \frac{a}{s^2+a^2}\right]$$

is the equivalence class in $\mathcal{E}^*$ containing the function

$$f: t \longmapsto e^{at} + \sin at \qquad (t \in [0, \infty)).$$

This equivalence class consists of all real functions with domain $[0, \infty)$ which map only a finite set of points t in each finite sub-interval of $[0, \infty)$ to numbers other than $f(t)$.

(ii) Since $h(s) = 0$ for $s \geqslant b$, h is in the equivalence class in $\mathcal{F}^*$ which also contains the zero function. Since, by Lerch's theorem, $\mathcal{L}$ is a one-to-one transformation from $\mathcal{E}^*$ into $\mathcal{F}^*$, it follows that $\mathcal{L}^{-1}[h]$ is the zero equivalence class in $\mathcal{E}^*$. In other words, $\mathcal{L}^{-1}[h]$ is the equivalence class of all functions with domain $[0, \infty)$, which map only a finite set of points in each finite sub-interval of $[0, \infty)$ to numbers other than zero. The members of this equivalence class are the null functions (see page K332).

29.1.6 Summary of Section 29.1

In this section we defined the terms

Theorems

1. (5-1, page K181)

If f is a piecewise continuous function of exponential order, there exists a real number α such that

$$\int_0^\infty e^{-st} f(t)\, dt$$

converges for all values of $s > \alpha$.

2. (page C11)

If n is a non-negative integer and a is a positive real number, then

$$\lim_{t \to \infty} t^n e^{-at} = 0.$$

3. (5-2, page K185) (Lerch's Theorem)

Let f and g be piecewise continuous functions of exponential order, and suppose there exists a real number s_0 such that

$$\mathcal{L}[f](s) = \mathcal{L}[g](s)$$

for all $s > s_0$. Then, with the possible exception of points of discontinuity,

$$f(t) = g(t)$$

for all $t > 0$.

4. (5-3, page K185)

If f is a function of exponential order, then

$$\lim_{s \to \infty} \mathcal{L}[f] = 0.$$

Techniques

1. Find the abscissa of convergence of $\mathcal{L}[f]$ for suitable $f \in \mathcal{F}$.

2. Find $\mathcal{L}[f]$ for suitable $f \in \mathcal{F}$.

Notation

$\mathcal{L}[f](s)$	(page K179)
$\mathcal{E}$	(page C13)
$\mathcal{F}$	(page C13)
$\mathcal{F}^*$	(page K184)
$\mathcal{E}^*$	(page C17)

ι

29.2   SOME TECHNIQUES

29.2.0   Introduction

We have seen that calculating the Laplace transform of a given function f is a fairly straightforward procedure: one has to evaluate the integral

$$\int_0^\infty e^{-st} f(t)\, dt.$$

Obtaining the inverse transform of a given function ϕ is a more difficult matter; in principle, it amounts to solving an integral equation, that is, finding a function f such that

$$\phi(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad (s \in (s_0, \infty)).$$

In practice, however, various techniques are available which simplify this problem. In this way Laplace transforms become an efficient method for solving initial-value problems. In this section we discuss some of the techniques which are useful in inverting Laplace transforms. The situation is very similar to that which arises when we proceed from formulas for differentiation of functions to formulas for integration. We set up a table of derived functions, with as many functions as we care to list in the left-hand column, and the derived functions in the right-hand column. If the function we wish to integrate can be expressed as a linear combination of functions that appear in the right-hand column of the table (e.g. if it is a polynomial), then the integration is easy. If not, try the techniques of integration by parts and by substitution, to knock the function into shape. It might work, or it might not.

We are in a similar position with regard to $\mathcal{L}$ and $\mathcal{L}^{-1}$. If we wish to calculate $\mathcal{L}^{-1}[f]$ for some f, the first thing to do is to try to express f as a linear combination of functions that we know to be images under $\mathcal{L}$. If this fails, there are methods, analogous to integration by parts and by substitution, which extend the range of functions f which can be dealt with.

29.2.1   A Table of Laplace Transforms

The first step is to list some common Laplace transforms.

READ Section 5-4 from page K186, *to line 3 of page* K187.

Note

Equations (5-7) to (5-10), page K186. We have derived all these in Section 29.1.

The table of transforms on pages K228-9 is important and will serve as a handy summary of the main properties of Laplace transforms. Ignore lines 12 and 13 on page K229 unless you studied the delta function in the optional part of *Unit 12, Linear Functionals and Duality*. With this table, we will solve a simple differential equation by the Laplace transform.

Example

Find the solution of

$$y'(t) - ay(t) = e^{-at} \qquad t \in [0, \infty)$$

with the initial condition $y(0) = 0$. We will use the result quoted at the beginning of this unit, that if $y(0) = 0$, then

$$\mathcal{L}[y'](s) = s\mathcal{L}[y](s).$$

Its proof is given in the next sub-section.

Taking the Laplace transforms of both sides of the differential equation, we find (using the linearity of $\mathcal{L}$)

$$\mathcal{L}[y'](s) - a\mathcal{L}[y](s) = \mathcal{L}[e^{-ar}](s)$$

i.e.

$$s\mathcal{L}[y](s) - a\mathcal{L}[y](s) = \frac{1}{s+a}$$

$$(s-a)\mathcal{L}[y](s) = \frac{1}{s+a}.$$

Solving this algebraic equation we find

$$\mathcal{L}[y](s) = \frac{1}{(s-a)(s+a)}$$

$$= \frac{1}{s^2 - a^2}$$

$$= \frac{1}{a}\left(\frac{a}{s^2 - a^2}\right)$$

$$= \frac{1}{a}\mathcal{L}[\sinh at](s)$$

by line 10 of the table on page K229.

Thus

$$y(t) = \frac{1}{a}\sinh at,$$

by Lerch's theorem (since y must be continuous).

Exercises

1. Using the table on page K229, find:

(i) $\mathcal{L}^{-1}\left[\dfrac{a}{s} + \dfrac{b}{s^2} + \dfrac{c}{s^3}\right]$

(ii) $\mathcal{L}^{-1}\left[\dfrac{1}{s-1} + \dfrac{2}{s-2} + \dfrac{3}{s-3}\right]$

giving your answers as continuous functions.

2. Use the Laplace transform and the table on page K229, to solve

$$y'(t) - ay(t) = e^{at}$$

$$y(0) = 0.$$

Check your solution.

Solutions

1. (i) $\mathcal{L}^{-1}\left[\dfrac{a}{s} + \dfrac{b}{s^2} + \dfrac{c}{s^3}\right] = a\mathcal{L}^{-1}\left[\dfrac{1}{s}\right] + b\mathcal{L}^{-1}\left[\dfrac{1}{s^2}\right] + c\mathcal{L}^{-1}\left[\dfrac{1}{s^3}\right]$

$$= a \times 1 + b \times t + c \times \frac{t^2}{2}$$

by the formula for $\mathcal{L}[t^n]$.

(ii) $\mathcal{L}^{-1}\left[\dfrac{1}{s-1} + \dfrac{2}{s-2} + \dfrac{3}{s-3}\right]$

$$= \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]$$

$$= e^t + 2e^{2t} + 3e^{3t}$$

by the formula for $\mathcal{L}[e^{at}]$

2. The Laplace transform of the differential equation is

$$\mathcal{L}[y'] - a\mathcal{L}[y] = \mathcal{L}[t \longmapsto e^{at}]$$

i.e.

$$s\mathcal{L}[y](s) - a\mathcal{L}[y](s) = \frac{1}{s-a}$$

since

$$\mathcal{L}[y'](s) = s\mathcal{L}[y](s)$$

if $y(0) = 0$.

Solving this algebraic equation gives

$$\mathcal{L}[y](s) = \frac{1}{(s-a)^2}$$

and so

$$y(t) = te^{at}$$

by the formula for $\mathcal{L}\left[\dfrac{t^{n-1}e^{at}}{(n-1)!}\right]$ with $n = 2$.

Check $y'(t) = (1 + at)e^{at}$;

so $y(0) = 0$ and $y'(t) - ay(t) = e^{at}$ as required.

29.2.2   The Differentiation Formula

In the example in the preceding sub-section, we solved a first-order differential equation with $y(0) = 0$ using the formula $\mathcal{L}[y'](s) = s\mathcal{L}[y](s)$. We may, however, want to solve equations with $y(0) \neq 0$, and equations of higher order than the first (for which additional initial conditions will be needed in order to give a unique solution). The next reading passage gives the generalization of the formula

$$\mathcal{L}[y'](s) = s\mathcal{L}[y](s)$$

which we need to deal with such problems.

READ from line 4 of page K187 *to line* −6 *of page* K189.

Notes

(i)   *line* −7, *page* K187   "vanishes at its upper limit" Since f is of exponential order we have

$$\lim_{t_0 \to \infty} e^{-st_0} f(t_0) \leqslant \lim_{t_0 \to \infty} e^{-st_0} Ce^{\alpha t_0}$$

$$= 0 \text{ if } s > \alpha$$

and similarly

$$\lim_{t_0 \to \infty} e^{-st_0} f(t_0) \geqslant -\lim_{t_0 \to \infty} e^{-st_0} Ce^{\alpha t} = 0.$$

Therefore

$$\lim_{t_0 \to \infty} e^{-st_0} f(t_0) = 0 \text{ as stated.}$$

(ii)   *line* −6, *page* K187   "jump discontinuity" was defined in *Unit 22* (page K330).

(iii)   *line* 9, *page* K189   We shall discuss the method of partial fractions in sub-section 29.2.5. To follow the example, all you need to do is to recognize that

$$\frac{1}{s-1} - \frac{1}{s} = \frac{1}{s(s-1)} \qquad (s \in R, s \neq 0, s \neq 1).$$

(iv)   *line* −10, *page* K189   "... of exponential order" What K seems to be saying here is that the study of constant-coefficient linear differential equations shows that *all* their solutions are linear combinations of functions of the type listed on

page K180, line −5. This is certainly true as far as the general solution of a *homogeneous* constant-coefficient equation is concerned. The situation for nonhomogeneous constant-coefficient equations is a little more involved. It can be shown that if $p(D)$ is any constant-coefficient linear differential operator and h is of exponential order, then every solution y of the differential equation

$$p(D)y = h$$

is of exponential order, but we shall not do this here.

(v) *line* −8 *page* K189 "... unique *continuous* solution." Since any solution of the differential equation must be at least n times differentiable, it is certainly continuous. Since the differential equation implies $\mathcal{L}[y] = \phi$, any such continuous solution must be a member of $\mathcal{L}^{-1}[\phi]$, and then Lerch's theorem shows that this continuous solution is unique.

Exercises

1. Solve

$$y'(t) - ay(t) = e^{-at} \qquad t \in [0, \infty)$$

   with the initial condition $y(0) = 1$.

2. Given that

$$y''(t) + y'(t) + y(t) = \sin 3t \qquad t \in [0, \infty)$$

   with $y'(0) = 4$, $y(0) = 5$, find $\mathcal{L}[y]$.

Solutions

1. Taking the Laplace transform and using Theorem 5-4 we obtain

$$\mathcal{L}[y'](s) - a\mathcal{L}[y](s) = \frac{1}{s+a}$$

   i.e.

$$s\mathcal{L}[y](s) - 1 - a\mathcal{L}[y](s) = \frac{1}{s+a}.$$

   Solving for $\mathcal{L}[y](s)$ gives

$$(s - a)\mathcal{L}[y](s) = \frac{1}{s+a} + 1$$

$$\mathcal{L}[y](s) = \frac{1}{s-a}\left(\frac{1}{s+a} + 1\right)$$

$$= \frac{1}{s^2 - a^2} + \frac{1}{s-a}$$

   i.e.

$$\mathcal{L}[y] = \frac{1}{a}\mathcal{L}[\sinh at] + \mathcal{L}[e^{at}]$$

$$= \mathcal{L}\left[\frac{1}{a}\sinh at + e^{at}\right]$$

   Thus

$$y(t) = \frac{1}{a}\sinh at + e^{at}.$$

2. The Laplace transform of the differential equation is

$$\mathcal{L}[y''](s) + \mathcal{L}[y'](s) + \mathcal{L}[y](s) = \frac{3}{s^2 + 9}.$$

Theorem 5-4 gives

$$(s^2 \mathcal{L}[y](s) - 5s - 4) + (s\mathcal{L}[y](s) - 5) + \mathcal{L}[y](s)$$

$$= \frac{3}{s^2 + 9}$$

and so

$$\mathcal{L}[y](s) = \frac{1}{s^2 + s + 1}\left(5s + 9 + \frac{3}{s^2 + 9}\right).$$

Notice how the manipulation of arbitrary constants has been replaced by the process of looking up inverse Laplace Transforms in a table.

29.2.3 The First Shifting Theorem

READ Section 5-5, from the beginning on page K193 to line 10 of page K194.

Notes

(i) *Equation (5-21), page K193* This is the most useful form of the first shifting theorem from the point of view of calculating images under $\mathcal{L}^{-1}$; to calculate images under $\mathcal{L}$, use Equation (5-20). Equation (5-21) tells us what happens if we shift the graph to the left or to the right in the codomain of $\mathcal{L}$.

(ii) *line 2, page K194* The technique of *completing the square* is very useful in dealing with quadratic polynomials that have no real factors. (See, for example, page K134, line 4; or Section III.2.3 of TI.)

In general, the method is to write $as^2 + bs + c$, where $b^2 < 4ac$, in the form

$$a\left[\left(s + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

Exercises

1. Using the first shifting theorem and the table on page K229, calculate

   (i) $\mathcal{L}[e^t \sin 2t]$

   (ii) $\mathcal{L}[e^{-t} t^2]$

   (iii) $\mathcal{L}[e^{-2t} \frac{t}{2a} \sin at]$.

2. Calculate

   (i) $\mathcal{L}^{-1}\left[\dfrac{1}{(s+2)^2 + 1}\right]$,   (ii) $\mathcal{L}^{-1}\left[\dfrac{s}{s^2 + 4s + 5}\right]$,

   (iii) $\mathcal{L}^{-1}\left[\dfrac{1}{4(s-1)^2 + 1}\right]$,   (iv) $\mathcal{L}^{-1}\left[\dfrac{1}{9s^2 - 6s + 2}\right]$.

Solutions

1. (i) Since

   $$\mathcal{L}[\sin 2t] = \frac{2}{s^2 + 4}$$

   (page K229), we have

   $$\mathcal{L}[e^t \sin 2t] = \frac{2}{(s - 1)^2 + 4},$$

   by Equation (5-20).

   (ii) Since

   $$\mathcal{L}[t^2] = \frac{2}{s^3},$$

we have

$$\mathcal{L}[e^{-t}t^2] = \frac{2}{(s+1)^3}.$$

(iii)   Since

$$\mathcal{L}\left[\frac{t}{2a}\sin at\right] = \frac{s}{(s^2+a^2)^2},$$

we have

$$\mathcal{L}\left[e^{-2t}\frac{t}{2a}\sin at\right] = \frac{s+2}{[(s+2)^2+a^2]^2}.$$

2.   (i) $\mathcal{L}^{-1}\left[\dfrac{1}{(s+2)^2+1}\right] = e^{-2t}\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right],$

by Equation (5-21),

$$= e^{-2t}\sin t \text{ (page K229)}$$

(ii) $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+4s+5}\right] = \mathcal{L}^{-1}\left[\dfrac{s}{(s+2)^2+1}\right]$

$$= e^{-2t}\mathcal{L}^{-1}\left[\frac{s-2}{s^2+1}\right]$$

$$= e^{-2t}(\cos t - 2\sin t)$$

(iii) $\mathcal{L}^{-1}\left[\dfrac{1}{4(s-1)^2+1}\right] = e^{t}\mathcal{L}^{-1}\left[\dfrac{1}{4s^2+1}\right]$

$$= \tfrac{1}{4}e^{t}\mathcal{L}^{-1}\left[\frac{1/2}{s^2+1/4}\right]$$

$$= \tfrac{1}{2}e^{t}\sin \tfrac{1}{2}t$$

(iv) $\mathcal{L}^{-1}\left[\dfrac{1}{9s^2-6s+2}\right] = \mathcal{L}^{-1}\left[\dfrac{1}{(3s-1)^2+1}\right]$

$$= e^{(1/3)t}\mathcal{L}^{-1}\left[\frac{1}{9s^2+1}\right]$$

$$= \tfrac{1}{9}e^{(1/3)t}\mathcal{L}^{-1}\left[\frac{\tfrac{1}{3}}{s^2+(\tfrac{1}{3})^2}\right]$$

$$= \tfrac{1}{3}e^{(1/3)t}\sin \tfrac{1}{3}t$$

(*Further practice*: Exercises 1, 2, 15, 31, 32 on pages K200-1.)

29.2.4   The Second Shifting Theorem (Optional)

Although this sub-section is optional we would encourage you to read it unless you are very short of time.

READ from line 11, page K194 *to line 2, page* K197, *and then from line* −5, *page* K198 *to the end of page* K199.

Notes

(i)   *Equation (5-23), page* K194   We can think of the unit step function as follows. If the domain of a function *f* includes points $t < 0$, their images under *f* are irrelevant to the Laplace transform. We can imagine the integral in the definition of $\mathcal{L}$ to be over the whole of $\mathcal{R}$, provided we first multiply *f* by u_0, because

$$\int_0^{\infty} k(t)\,dt = \int_{-\infty}^{\infty} k(t)u_0(t)\,dt$$

for any function $k: \mathscr{R} \longrightarrow \mathscr{R}$. In particular

$$\mathfrak{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt - \int_{-\infty}^\infty e^{-st} f(t) u_0(t)\, dt.$$

More generally:

$$\int_a^\infty k(t)\, dt = \int_{-\infty}^\infty k(t) u_a(t)\, dt.$$

Thus the unit step function is a device for altering an end-point in an "infinite" integral.

(ii) *Equation (5-26), page* K195 You may find the following form of Equation (5-26) easier to use:

$$\mathfrak{L}[f] = e^{-as}\mathfrak{L}[t \longmapsto f(t+a)]$$

provided $f(t) = 0$ for $t < a$. It tells us what happens if we shift the graph to the left or to the right in the domain of $\mathfrak{L}$.

(iii) *Equation (5-27), page* K196 This is the most useful form of the second shifting theorem from the point of view of calculating inverse images.

(iv) *Example 4, page* K196 The function f has a different formula in the three intervals $[0, 1]$, $(1, 2]$ and $(2, \infty)$. We want functions f_1, f_2, f_3 such that

$$f = f_1 + f_2 + f_3,$$

and the f_i have formulas simple enough for their Laplace transforms to be written down directly. We achieve this by making

$$f(t) = f_1(t) + f_2(t) + f_3(t)$$

$$t \in [0, 1] \qquad 1 = f_1(t) + 0 \quad + 0$$

$$t \in (1, 2] \quad 2 - t = f_1(t) + f_2(t) + 0$$

$$t \in (2, \infty) \qquad 0 = f_1(t) + f_2(t) + f_3(t)$$

To satisfy the equation for $t \in [0, 1]$ we put $f_1(t) = 1$, and $f_2(t) = f_3(t) = 0$. Then we satisfy the next equation by keeping $f_1(t) = 1$, $f_3(t) = 0$ and by putting in the appropriate formula for $f_2(t)$, namely $1 - t$. Finally, to satisfy the last equation, we put $-(2 - t)$ for $f_3(t)$, and keep $f_1(t) = 1$, $f_2(t) = 1 - t$.

(v) *line* 13, *page* K199 "the sum of the geometric series. . . ." The series converges only if $s > 0$, and so the domain of $\mathfrak{L}[f]$ is $\mathscr{R}^+$.

Exercises

1. Use the second shifting theorem and the table on page K229 to calculate $\mathfrak{L}[f]$ where

   (i) $f(t) = u_\pi(t) \cos t$

   (ii) $f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 2 - t & \text{if } 1 < t < 2 \\ 0 & \text{if } t \geqslant 2 \end{cases}$
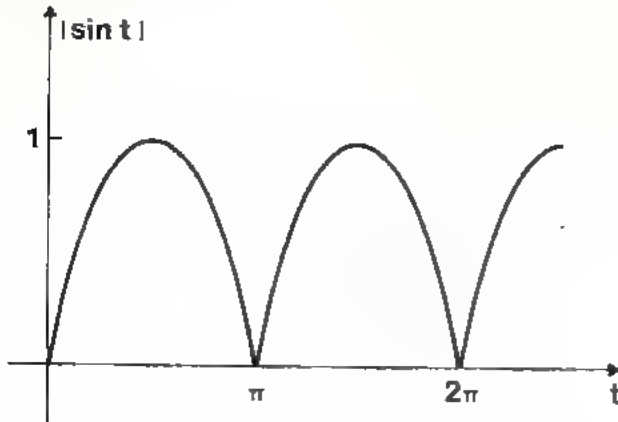
   (*Hint*: see Example 4, page K196)

   (iii) $f(t) = \begin{cases} \sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } t \geqslant \pi \end{cases}$

2. Calculate

   (i) $\mathfrak{L}^{-1}\left[\dfrac{e^{-s}}{s+1} \right]$

   (ii) $\mathfrak{L}^{-1}\left[\dfrac{e^{-3s}}{(s+1)^2 + 1} \right]$

3. Calculate $\mathcal{L}[|\sin t|]$ by using Theorem 5-9. (The function $t \longmapsto |\sin t|$ is the "rectified sine wave" which is also discussed in *Unit 22, Fourier Series*.)

Solutions

1.   (i) $\mathcal{L}[u_\pi(t) \cos t](s) = e^{-\pi s}\mathcal{L}[\cos(t + \pi)](s)$ (Equation (5-26))

$$= e^{-\pi s}\mathcal{L}[-\cos t](s)$$

$$= -\frac{e^{-\pi s}s}{s^2 + 1} \qquad \text{(page K229)}$$

  (ii) $f(t) = f_1(t) + f_2(t) + f_3(t)$

where

$$f_1(t) = tu_0(t)$$

$$f_2(t) = (2 - 2t)u_1(t)$$

$$f_3(t) = (-2 + t)u_2(t).$$

Hence

$$\mathcal{L}[f] = \mathcal{L}[f_1] + \mathcal{L}[f_2] + \mathcal{L}[f_3]$$

$$\mathcal{L}[f](s) = \frac{1}{s^2} + e^{-s}\mathcal{L}[2 - 2(t + 1)](s)$$

$$+ e^{-2s}\mathcal{L}[-2 + (t + 2)](s)$$

(Equation (5-26))

$$= \frac{1}{s^2} + e^{-s}\left(\frac{-2}{s^2}\right) + e^{-2s}\left(\frac{1}{s^2}\right) \quad \text{(page K229)}$$

$$= \frac{1 - 2e^{-s} + e^{-2s}}{s^2}$$

$$= \left(\frac{1 - e^{-s}}{s}\right)^2.$$

  (iii) $f(t) = f_1(t) + f_2(t)$

where

$$f_1(t) = \sin t$$

$$f_2(t) = -u_\pi(t) \sin t.$$

So

$$\mathcal{L}[f] = \mathcal{L}[\sin t] + \mathcal{L}[-u_\pi(t) \sin t]$$

$$= \mathcal{L}[\sin t] + e^{-\pi s}\mathcal{L}(-\sin (t + \pi))$$

$$= \mathcal{L}[\sin t] + e^{-\pi s}\mathcal{L}[\sin t]$$

i.e.,

$$\mathcal{L}[f](s) = (1 + e^{-\pi s})\frac{1}{s^2 + 1}.$$

2.   (i)   Since

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t} \quad \text{(page K229)},$$

we have

$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s+1}\right] = u_1(t)e^{-(t-1)}$$

$$\text{(Equation (5-27))}.$$

   (ii)   Since

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t \quad \text{(page K229)},$$

we have

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}\sin t$$

$$\text{(Equation (5-21))}$$

and so

$$\mathcal{L}^{-1}\left[\frac{e^{-3s}}{(s+1)^2+1}\right] = u_3(t)e^{3-t}\sin(t-3)$$

$$\text{(Equation (5-27))}.$$

3.   Theorem 5-9 gives

$$(1 - e^{-\pi s})\,\mathcal{L}[|\sin t|](s) = \int_{\blacksquare}^{\pi} e^{-st}\sin t\,dt$$

$$= \left[\frac{e^{-st}(-s\sin t - \cos t)}{s^2+1}\right]_0^{\pi}$$

by Section III.5.2 (ii) of **TI**

$$= \frac{e^{-s\pi}+1}{s^2+1}$$

so that

$$\mathcal{L}[|\sin t|] = \left(\frac{1+e^{-s\pi}}{1-e^{-s\pi}}\right)\left(\frac{1}{s^2+1}\right).$$

29.2.5   Partial Fractions

Earlier in the unit (page K189) we evaluated an inverse Laplace transform by making use of the equation

$$\frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s} \tag{1}$$

which expressed (in abbreviated form) the function on the left as a linear combination of two functions whose inverse transforms are tabulated. The table of transforms on page K229 lists only a limited number of cases, but any function of the form $\frac{p}{q}$, where p and q are polynomials such that the degree of p is less than that of q can be reduced to a linear combination of the listed cases by the method of *partial fractions*. We consider, in place of $s(s-1)$, any polynomial q whose n roots are all real and distinct:

$$q(s) = (s - a_1)(s - a_2)\cdots(s - a_n).$$

Then in place of the linear combination on the right-hand side of (1) we can consider the linear combination of the functions

$$\frac{1}{s-a_1}, \frac{1}{s-a_2}, \ldots, \frac{1}{s-a_n} \quad (s \in R, s \neq a_1, \ldots, a_n) \tag{2}$$

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This leads us to consider whether *all* functions of the form

$$\frac{(\text{polynomial of degree} \leqslant n-1)}{q(s)} \tag{3}$$

can be constructed in this way. In fact they can, as the following theorem shows.

Theorem

For a given polynomial function q, the set V of all functions of the form (3) is a vector space of dimension n, and if q is of degree n and has n distinct roots $a_1, \ldots, a_n$, then the set of n functions listed in (2) is a basis for V.

Proof

The first part of the theorem follows from the fact that the set of all polynomial functions of degree $\leqslant n-1$ is a vector space of dimension n. For the second part, it is sufficient to prove that the n functions listed in (2) are linearly independent elements of V. Suppose that

$$\frac{\alpha_1}{s - a_1} + \cdots + \frac{\alpha_n}{s - a_n}$$

is the zero function. We find on multiplying by

$$(s - a_1)(s - a_2) \ldots (s - a_n)$$

that

$$\alpha_1 (s - a_2) \ldots (s - a_n)$$
$$+ (s - a_1)\alpha_2 (s - a_3) \ldots (s - a_n)$$
$$\cdots + (s - a_1) \ldots (s - a_{n-1})\alpha_n \qquad (s \in R, s \neq a_1, \ldots, a_n)$$

is the zero function on this domain. Although a_1 is not in the domain, we can take the limit as s approaches a_1; this limit is

$$\alpha_1 (a_1 - a_2) \ldots (a_1 - a_n),$$

and since $a_1, \ldots, a_n$ are all different we conclude that $\alpha_1 = 0$. Similarly we can show that $\alpha_2 = 0, \ldots, \alpha_n = 0$, and hence that the n functions listed in (2) are linearly independent. By Theorem 3.3 on page N16, they therefore constitute a basis.

Example

Express $\dfrac{s + 4}{s^2 + 5s + 6}$ as a sum of partial fractions.

We first factorize $s^2 + 5s + 6$, obtaining $(s + 2)(s + 3)$. A basis for the space V of all functions of the form

$$\frac{(\text{polynomial of degree} \leqslant 1)}{s^2 + 5s + 6}$$

is $\left\{ \dfrac{1}{s + 2}, \dfrac{1}{s + 3} \right\}$. We therefore have, for some real α_1 and α_2,

$$\frac{s + 4}{s^2 + 5s + 6} = \frac{\alpha_1}{s + 2} + \frac{\alpha_2}{s + 3}$$

i.e.

$$\frac{s + 4}{s^2 + 5s + 6} = \frac{\alpha_1(s + 3) + \alpha_2(s + 2)}{(s + 2)(s + 3)}.$$

Equating the coefficients of s and 1 in the numerators (which we may do since $\{s, 1\}$ is a basis for the vector space of all polynomials of degree $\leqslant 1$), we obtain

$$1 = \alpha_1 + \alpha_2$$

and

$$4 = 3\alpha_1 + 2\alpha_2$$

so that $\alpha_1 = 2$, $\alpha_2 = -1$, and the *partial fraction expansion* is

$$\frac{s+4}{s^2 + 5s + 6} = \frac{2}{s+2} - \frac{1}{s+3}.$$

The method can be generalized to cases where the nth degree polynomial q does not have n distinct linear factors. As before, let V stand for the set of all functions of the form

$$\frac{\text{polynomial of degree} \leqslant n-1}{q}$$

where n is the degree of q. Then V has a basis consisting of the following elements:

(a)   for each distinct linear factor of the form $s - a$ in $q(s)$, there is an element

$$\frac{1}{s-a}$$

in the basis;

(b)   for each factor of the form $(s-a)^k$ in $q(s)$, there are k elements

$$\frac{1}{s-a}, \frac{1}{(s-a)^2}, \ldots, \frac{1}{(s-a)^k}$$

in the basis;

(c)   for each factor of the form $s^2 + as + b$ in $q(s)$, with $a^2 < 4b$ (so that $s^2 + as + b$ has no real linear factors), there are two elements

$$\frac{1}{s^2 + as + b}, \frac{s}{s^2 + as + b}$$

in the basis;

(d)   for each factor of the form $(s^2 + as + b)^k$ in $q(s)$, with $a^2 < 4b$, there are $2k$ elements

$$\frac{1}{s^2 + as + b}, \frac{1}{(s^2 + as + b)^2}, \ldots, \frac{1}{(s^2 + as + b)^k}$$

$$\frac{s}{s^2 + as + b}, \frac{s}{(s^2 + as + b)^2}, \ldots, \frac{s}{(s^2 + as + b)^k}$$

in the basis.

We shall not prove this result.

Example

To express $\dfrac{s-1}{(s+1)(s^2+1)}$ in partial fractions, we use rules (a) and (c) above.

This gives

$$\frac{s-1}{(s+1)(s^2+1)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{s^2+1} + \frac{\alpha_3 s}{s^2+1}$$

$$= \frac{\alpha_1(s^2+1) + (\alpha_2 + \alpha_3 s)(s+1)}{(s+1)(s^2+1)}.$$

Equating the coefficients of s^2, s, 1 in the numerators, we obtain

$$\alpha_1 \qquad + \alpha_3 = 0$$

$$\alpha_2 + \alpha_3 = 1$$

$$\alpha_1 + \alpha_2 \qquad = -1$$

Thus

$$\alpha_1 = -1, \alpha_2 = 0, \alpha_3 = 1.$$

Example

Find $\mathcal{L}^{-1}\left[\dfrac{s^2 + s + 1}{(s + 2)^2(s + 3)}\right]$ by using partial fractions. Applying rules (a) and (b) above, we look for $\alpha_1, \alpha_2, \alpha_3$ such that

$$\frac{s^2 + s + 1}{(s + 2)^2(s + 3)} = \frac{\alpha_1}{s + 2} + \frac{\alpha_2}{(s + 2)^2} + \frac{\alpha_3}{(s + 3)}$$

$$= \frac{\alpha_1(s + 2)(s + 3) + \alpha_2(s + 3) + \alpha_3(s + 2)^2}{(s + 2)^2(s + 3)}.$$

Equating coefficients of s^2, s and 1 in the numerators, and solving the resulting system of three simultaneous equations for $\alpha_1, \alpha_2, \alpha_3$, we obtain

$$\alpha_1 = -6, \alpha_2 = 3, \alpha_3 = 7.$$

The required inverse Laplace transform is therefore

$$\mathcal{L}^{-1}\left[\frac{s^2 + s + 1}{(s + 2)^2(s + 3)}\right] = \mathcal{L}^{-1}\left[\frac{-6}{s + 2} + \frac{3}{(s + 2)^2} + \frac{7}{(s + 3)}\right]$$

$$= -6e^{-2t} + 3te^{-2t} + 7e^{-3t}.$$

The type of function that we can expand in partial fractions—the ratio of two polynomials—is called a *rational function*. Here we have only considered the case where the polynomial in the numerator has degree less than the one in the denominator. For Laplace transforms this case is the only one that arises, because of Theorem 5-3 on page K185.

Alternative Method

In the case where the denominator $q(s)$ of the rational function has degree n and has n distinct linear factors, the calculation of the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ in

$$\frac{p(s)}{q(s)} = \sum_{i=1}^{n} \frac{\alpha_i}{s - a_i}$$

can be perfomed without solving a system of n linear equations, by means of the following theorem.

Theorem (Optional)

If

$$q(s) = (s - a_1) \ldots (s - a_n)$$

with all the a_i distinct, and $p(s)$ is of degree $\leqslant n - 1$, then in

$$\frac{p(s)}{q(s)} = \sum_{i=1}^{n} \frac{\alpha_i}{(s - a_i)}$$

each α_i is given by

$$\alpha_i = \frac{p(a_i)}{\underbrace{(a_i - a_1)(a_i - a_2) \ldots (a_i - a_n)}_{\text{omitting } (a_i - a_i)}}$$

(e.g.

$$\alpha_1 = \frac{p(a_1)}{(a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n)},$$

$$\alpha_n = \frac{p(a_n)}{(a_n - a_1)(a_n - a_2) \ldots (a_n - a_{n-1})}).$$

Proof

$$\lim_{s \to a_i} \left((s - a_i) \frac{p(s)}{q(s)} \right) - \sum_{j=1}^{n} \lim_{s \to a_i} \left(\frac{(s - a_i) \alpha_j}{s - a_j} \right)$$

$$= 0 + 0 + \cdots + 0 + \alpha_i + 0 + \cdots + 0.$$

On the other hand

$$\frac{(s - a_i) p(s)}{q(s)} - \underbrace{\frac{p(s)}{(s - a_1) \ldots (s - a_n)}}_{\text{omitting } (s - a_i)}.$$

This function is continuous at a_i, and so its limit is

$$\underbrace{\frac{p(a_i)}{(a_i - a_1) \ldots (a_i - a_n)}}_{\text{omitting } (a - a_i)}.$$

Exercises

1. Write down partial fraction expansions, containing undetermined numbers $\alpha_1, \alpha_2, \ldots$, for the following functions. Do not calculate $\alpha_1, \alpha_2, \ldots$.

    (i) $\dfrac{s}{s^2 - 1}$

    (ii) $\dfrac{1}{s(s + 2)^2}$

    (iii) $\dfrac{1}{s(s^2 + s - 2)}$

    (iv) $\dfrac{1}{s(s^2 + s + 2)}$

    (v) $\dfrac{s(s + 1)}{(s^2 + 1)^2}$

2. Calculate $\alpha_1, \alpha_2, \ldots$, for parts (i), (ii), (iv) of Exercise 1 and so obtain the appropriate partial fraction expansions. (If appropriate you can use the alternative method.)

3. Use the answers to Exercise 2, and the formulas on page K229, to calculate

    (i) $\mathcal{L}^{-1} \left[\dfrac{s}{s^2 - 1} \right]$,

    (ii) $\mathcal{L}^{-1} \left[\dfrac{1}{s(s + 2)^2} \right]$,

    (iii) $\mathcal{L}^{-1} \left[\dfrac{1}{s(s^2 + s + 2)} \right]$

Solutions

1.   (i). $\dfrac{s}{s^2 - 1} = \dfrac{s}{(s - 1)(s + 1)} = \dfrac{\alpha_1}{s - 1} + \dfrac{\alpha_2}{s + 1}$ (case (a))

    (ii) $\dfrac{1}{s(s + 2)^2} = \dfrac{\alpha_1}{s} + \dfrac{\alpha_2}{s + 2} + \dfrac{\alpha_3}{(s + 2)^2}$ (case (b))

    (iii) $\dfrac{1}{s(s^2 + s - 2)} = \dfrac{1}{s(s + 2)(s - 1)} = \dfrac{\alpha_1}{s} + \dfrac{\alpha_2}{s + 2} + \dfrac{\alpha_3}{s - 1}$

                                                             (case (a))

(iv) $\dfrac{1}{s(s^2 + s + 2)} = \dfrac{\alpha_1}{s} + \dfrac{\alpha_2}{s^2 + s + 2} + \dfrac{\alpha_3 s}{s^2 + s + 2}$

<div align="right">(case (c))</div>

(v) $\dfrac{s(s + 1)}{(s^2 + 1)^2} = \dfrac{\alpha_1}{s^2 + 1} + \dfrac{\alpha_2 s}{s^2 + 1} + \dfrac{\alpha_3}{(s^2 + 1)^2} + \dfrac{\alpha_4 s}{(s^2 + 1)^2}$

<div align="right">(case (d))</div>

2. (i) We have

$$\frac{s}{s^2 - 1} = \frac{\alpha_1}{s - 1} + \frac{\alpha_2}{s + 1}$$

Bringing the right-hand side to a common denominator and equating powers of s and 1 on both sides, we obtain

$$1 = \alpha_1(s + 1) + \alpha_2(s - 1)$$

i.e.

$$\alpha_1 + \alpha_2 = 1 \text{ and } \alpha_1 - \alpha_2 = 0$$

so that

$$\alpha_1 = \tfrac{1}{2}, \alpha_2 = \tfrac{1}{2}.$$

The expansion is therefore

$$\frac{s}{s^2 - 1} = \frac{\tfrac{1}{2}}{s - 1} + \frac{\tfrac{1}{2}}{s + 1}.$$

In this case $q(s)$ has distinct linear factors and so the alternative method can be used. Here

$$p(s) = s, \quad q(s) = (s - 1)(s + 1);$$

so

$$\alpha_1 = \frac{p(1)}{1 + 1} = \frac{1}{2}$$

and

$$\alpha_2 = \frac{p(-1)}{-1 - 1} = \frac{1}{2}$$

As before

$$\frac{s}{s^2 - 1} = \frac{\tfrac{1}{2}}{s - 1} + \frac{\tfrac{1}{2}}{s + 1}.$$

It is a good check to try a few special values of s; e.g. here $s = 2$ gives

$$\frac{2}{3} = \frac{1}{2 \times 1} + \frac{1}{2 \times 3}$$

(ii) Since $\dfrac{1}{s(s + 2)^2} = \dfrac{\alpha_1}{s} + \dfrac{\alpha_2}{s + 2} + \dfrac{\alpha_3}{(s + 2)^2}$,

we have

$$1 = \alpha_1(s + 2)^2 + \alpha_2 s(s + 2) + \alpha_3 s.$$

Equating coefficients of s^2, s, 1 gives

$$\alpha_1 + \alpha_2 \qquad\quad = 0$$
$$4\alpha_1 + 2\alpha_2 + \alpha_3 = 0$$
$$4\alpha_1 \qquad\qquad\quad = 1$$

whence $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \quad \frac{1}{4}$, $\alpha_3 = -\frac{1}{2}$, and

$$\frac{1}{s(s+2)^2} = \frac{1}{4s} - \frac{1}{4(s+2)} - \frac{1}{2(s+2)^2}$$

Check: $s = 1$ gives

$$\frac{1}{3^2} = \frac{1}{4} - \frac{1}{4 \times 3} - \frac{1}{2 \times 3^2} = \frac{9 - 3 - 2}{36}$$

(iv) Since

$$\frac{1}{s(s^2 + s + 2)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2 + s + 2} + \frac{\alpha_3 s}{s^2 + s + 2}$$

we have

$$1 = \alpha_1(s^2 + s + 2) + \alpha_2 s + \alpha_3 s^2$$

whence

$$\alpha_1 + \quad \alpha_3 = 0$$
$$\alpha_1 + \alpha_2 \quad = 0$$
$$2\alpha_1 \quad\quad = 1$$

so that

$$\frac{1}{s(s^2 + s + 2)} = \frac{1}{2s} - \frac{1}{2(s^2 + s + 2)}$$

$$- \frac{s}{2(s^2 + s + 2)}$$

Check: $s = 1$ gives $\dfrac{1}{4} = \dfrac{1}{2} - \dfrac{1}{2 \times 4} - \dfrac{1}{2 \times 4}$.

3. (i) $\mathcal{L}^{-1}\left[\dfrac{s}{s^2 - 1}\right] = \mathcal{L}^{-1}\left[\dfrac{1}{2(s-1)} + \dfrac{1}{2(s+1)}\right] = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$

(This one can also be obtained directly from page K229, as cosh t.)

(ii) $\mathcal{L}^{-1}\left[\dfrac{1}{s(s+2)^2}\right] = \frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}$

(iii) $\mathcal{L}^{-1}\left[\dfrac{1}{s(s^2 + s + 2)}\right]$

$$= \mathcal{L}^{-1}\left[\frac{1}{2s} - \frac{1}{2((s+\frac{1}{2})^2 + \frac{7}{4})} - \frac{(s+\frac{1}{2}) - \frac{1}{2}}{2((s+\frac{1}{2})^2 + \frac{7}{4})}\right]$$

$$= \frac{1}{2} - \frac{1}{2}e^{-t/2}\sqrt{\frac{4}{7}}\sin\sqrt{\frac{7}{4}}t$$

$$\quad - \frac{1}{2}e^{-t/2}(\cos\sqrt{\frac{7}{4}}t - \frac{1}{2}\sqrt{\frac{4}{7}}\sin\sqrt{\frac{7}{4}}t)$$

$$= \frac{1}{2} - \frac{1}{2\sqrt{7}}e^{-t/2}\sin\frac{\sqrt{7}}{2}t - \frac{1}{2}e^{-t/2}\cos\frac{\sqrt{7}}{2}t$$

29.2.6 Summary of Section 29.2

In this section we defined the terms

unit step function	(page K194)	* * *
partial fraction expansion	(page C30)	* *
rational function	(page C31)	* *

Theorems

1. (5-4, page K187)

Let f be continuous on $(0, \infty)$, and suppose that f' is piecewise continuous and of exponential order on $[0, \infty)$. Then

      * *

$$\mathcal{L}[f'] = s\mathcal{L}[f] - f(0^+)$$

where

$$f(0^+) = \lim_{t \to 0^+} f(t).$$

More generally, if $f, f', \ldots, f^{(n-1)}$ are continuous for all $t > 0$, and if $f^{(n)}$ is piecewise continuous and of exponential order on $[0, \infty)$, then

$$\mathcal{L}[f''] = s^2 \mathcal{L}[f] - sf(0^+) - f'(0^+)$$
$$\vdots$$
$$\mathcal{L}[f^{(n)}] = s^n \mathcal{L}[f] - s^{n-1} f(0^+) - s^{n-2} f'(0^+) - \cdots - f^{(n-1)}(0^+).$$

2. (5-6, page K193) (first shifting theorem)   * *

If $\mathcal{L}[f] = \phi(s)$, then

$$\mathcal{L}[e^{at} f(t)] = \phi(s - a).$$

3. (page C29)

For a given polynomial function q, the set V of all functions of the form   * *

$$\frac{(\text{polynomial of degree} \leqslant n - 1)}{q}$$

is a vector space of dimension n, and if q is of degree n and has n distinct roots $a_1, \ldots, a_n$ then the set of n functions

$$\frac{1}{s - a_1}, \frac{1}{s - a_2}, \ldots, \frac{1}{s - a_n} \qquad (s \in R, s \neq a_1, \ldots, a_n)$$

is a basis for V.

4. (page C31)

If

$$q(s) = (s - a_1) \cdots (s - a_n)$$

with all the a_i distinct, and $p(s)$ is of degree $\leqslant n - 1$, then in

$$\frac{p(s)}{q(s)} = \sum_{i=1}^{n} \frac{\alpha_i}{(s - a_i)}$$

each α_i is given by

$$\alpha_i = \frac{p(a_i)}{\underbrace{(a_i - a_1)(a_i - a_2) \cdots (a_i - a_n)}_{\text{omitting } (a_i - a_i)}}.$$

Techniques

1. Given the Laplace transform of a function, find the function.   * *

2. Use Laplace transforms to solve a constant-coefficient initial-value   * *
   problem.

3. Given a rational function, find its partial fraction expansion.   * *

Notation

$u_a(t)$ (page K194)

29.3 APPLICATIONS OF THE LAPLACE TRANSFORM

29.3.1 Differential Equations

We have already seen some simple examples of initial-value problems for differential equations solved using the Laplace transform. For an example of a more complicated equation,

READ Section 5-6, page K202 to line −10 on page K203.

Notes

(i) *lines 8–11, page K203*   This manipulation uses Theorem 5-8, which we have not asked you to read. The same result can be achieved without using the theorem, as follows:

$$\mathcal{L}^{-1}\left[\frac{3(s+2)}{(s^2+4s+13)^2}\right] = \mathcal{L}^{-1}\left[\frac{3(s+2)}{[(s+2)^2+9]^2}\right]$$

$$= 3e^{-2t}\mathcal{L}^{-1}\left[\frac{s}{(s^2+9)^2}\right]$$

$$= 3e^{-2t}\frac{t}{6}\sin 3t \qquad \text{(line −3, page K229)}$$

(ii) *line −13, page K203*   "The student who feels ..." A further advantage of the Laplace transform method is that it is very suitable for dealing with cases where the nonhomogeneous term of the differential equation has different formulas in different intervals, like the function illustrated in Fig. 5-7 on page K196.

The Laplace transform method is particularly useful for dealing with systems of differential equations. Consider, for example, the system

$$2y_1'(t) + y_2'(t) + y_1(t) + 2y_2(t) = 1$$
$$y_1'(t) + y_2'(t) + y_1(t) + y_2(t) = t$$

with $y_1(0) = 1$, $y_2(0) = 0$.

Since this system is not homogeneous, the methods we studied in *Unit 13, Systems of Differential Equations*, are insufficient to solve it. The Laplace transform method, on the other hand, yields the solution just as easily as for a single equation. The Laplace transforms of the equations are

$$(2s+1)\mathcal{L}[y_1](s) + (s+2)\mathcal{L}[y_2](s) = 2 + \frac{1}{s}$$

$$(s+1)\mathcal{L}[y_1](s) + (s+1)\mathcal{L}[y_2](s) = 1 + \frac{1}{s^2}$$

Solving this pair of algebraic equations for $\mathcal{L}[y_1](s)$ and $\mathcal{L}[y_2](s)$, we obtain

$$\mathcal{L}[y_1](s) = \frac{1}{s^2-1}\left(s+1-\frac{2}{s^2}\right) = \frac{s^2+2s+2}{s^2(s+1)}$$

$$\mathcal{L}[y_2](s) = \frac{1}{s^2-1}\left(-2+\frac{1}{s}+\frac{1}{s^2}\right) = \frac{2s-1}{s^2(s+1)}$$

In terms of partial fractions

$$\mathcal{L}[y_1](s) = \frac{1}{s+1} + \frac{2}{s^2} \qquad \mathcal{L}[y_2](s) = \frac{1}{s+1} - \frac{1}{s} - \frac{1}{s^2}$$

Thus

$$y_1 = \mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{2}{s^2}\right], \qquad \text{i.e. } y_1: t \longmapsto e^{-t} + 2t$$

and

$$y_2 = \mathcal{L}^{-1}\left[\frac{1}{s+1} - \frac{1}{s} - \frac{1}{s^2}\right], \qquad \text{i.e. } y_2: t \longmapsto e^{-t} - 1 - t$$

Exercises

1. Exercise 4, page K204.
   (Use the formula

   $$\frac{25s}{(s^2 + 1)(s - 2)^2} = -\frac{3}{s - 2} + \frac{10}{(s - 2)^2} + \frac{3s - 4}{s^2 + 1}.)$$

2. Use Laplace transforms to solve the system

   $$y_1' = \lambda y_1 + y_2$$
   $$y_2' = \lambda y_2 + y_3$$
   $$\vdots$$
   $$y_{n-1}' = \lambda y_{n-1} + y_n$$
   $$y_n' = \lambda y_n$$

   with $y_1(0) = \cdots = y_{n-1}(0) = 0$, $y_n(0) = c$. Here λ is some real number; n is a positive integer, and $y_1, \ldots, y_n$ are the unknown functions.

Solutions

1. Taking the Laplace transform of the differential equation we obtain (using Theorem 5-4)

   $$s^2 \mathcal{L}[y](s) - \tfrac{3}{25}s + \tfrac{4}{25} - 4(s\mathcal{L}[y](s) - \tfrac{3}{25}) + 4\mathcal{L}[y](s)$$

   $$= \frac{2}{s - 2} + \frac{s}{s^2 + 1}$$

   so that

   $$\mathcal{L}[y](s) = \frac{-16 + 3s}{25(s - 2)^2} + \frac{2}{(s - 2)^3} + \frac{s}{(s^2 + 1)(s - 2)^2}.$$

   The inverse transforms of the various terms are

   $$\mathcal{L}^{-1}\left[\frac{-16 + 3s}{25(s - 2)^2}\right] = e^{2t}\mathcal{L}^{-1}\left[\frac{-10 + 3s}{25s^2}\right] \text{ (Equation 5-22)}$$

   $$= e^{2t}\left(\frac{-10}{25}t + \frac{3}{25}\right)$$

   $$\mathcal{L}^{-1}\left[\frac{2}{(s - 2)^3}\right] = 2\left[\frac{t^2 e^{2t}}{2}\right] = t^2 e^{2t}$$

   $$\mathcal{L}^{-1}\left[\frac{s}{(s^2 + 1)(s - 2)^2}\right]$$

   $$= \frac{1}{25}\mathcal{L}^{-1}\left[\frac{-3}{s - 2} + \frac{10}{(s - 2)^2} + \frac{3s - 4}{s^2 + 1}\right]$$

   $$= \frac{-3}{25}e^{2t} + \frac{10}{25}te^{2t} - \frac{4}{25}\sin t + \frac{3}{25}\cos t$$

   Therefore

   $$y(t) = e^{2t}\left(-\frac{10}{25}t + \frac{3}{25}\right) + t^2 e^{2t} - \frac{3}{25}e^{2t} + \frac{10}{25}te^{2t}$$

   $$- \frac{4}{25}\sin t + \frac{3}{25}\cos t$$

   $$= t^2 e^{2t} - \frac{4}{25}\sin t + \frac{3}{25}\cos t$$

2. For $i = 1, 2, \ldots, n - 1$,

   $$\mathcal{L}[y_i'] = \lambda\mathcal{L}[y_i] + \mathcal{L}[y_{i+1}]$$

   i.e.

   $$(s - \lambda)\mathcal{L}[y_i](s) = \mathcal{L}[y_{i+1}](s).$$

The nth equation yields

$$(s - \lambda)\mathcal{L}[y_n](s) = c.$$

It follows immediately by applying the previous equation with $i = n - 1$, then $n - 2$, etc. that

$$\mathcal{L}[y_i](s) = \frac{c}{(s - \lambda)^{n-i+1}}, \quad i - 1, \ldots, n,$$

and

$$y_i(t) = \frac{ct^{n-i}e^{\lambda t}}{(n - i)!}.$$

This is an extremely neat method of obtaining the result. Compare it with the one used in sub-section 13.2.2 of *Unit 13* which involves numerous arbitrary constants.

29.3.2 Electrical Engineering (Optional)

As we have seen, the great virtue of the Laplace transform is that it enables us to replace constant-coefficient differential equations by algebraic equations. This is particularly useful in electrical engineering, since it makes it possible to analyse networks containing inductors and capacitors by elementary methods instead of solving differential equations. For example, to analyse the network shown, the method described in *Unit 4, Differential Equations I*, and *Unit 11, Differential Equations II* (page K170), leads us to formulate the following differential equations.
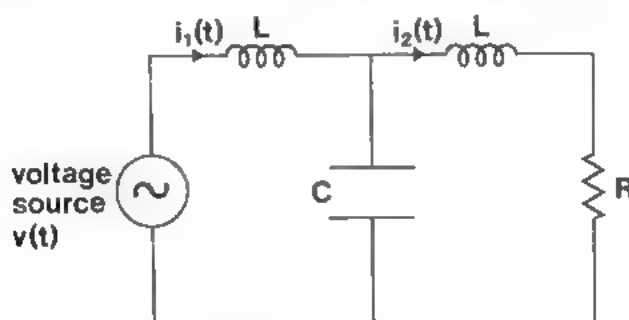
Kirchhoff's first law

$$\frac{dq}{dt} = i_1(t) - i_2(t) \qquad t \geqslant 0$$

Kirchhoff's second law

$$L\frac{di_1}{dt} + \frac{1}{C}q(t) = v(t) \quad t \geqslant 0 \text{ (left-hand loop)}$$

$$L\frac{di_2}{dt} + Ri_2(t) - \frac{1}{C}q(t) = 0 \quad t \geqslant 0 \text{ (right-hand loop)} \tag{1}$$

We then have a third-order system of equations to solve. If the initial values of i_1, i_2 and q are 0, we may solve the equations by taking the Laplace transform as follows. Putting

$$Q = \mathcal{L}[q], \ I_1 = \mathcal{L}[i_1], \ I_2 = \mathcal{L}[i_2], \ V = \mathcal{L}[v],$$

we get

$$sQ(s) = I_1(s) - I_2(s)$$

$$sLI_1(s) + \frac{1}{C}Q(s) = V(s) \tag{2}$$

$$sLI_2(s) + RI_2(s) - \frac{1}{C}Q(s) = 0$$

Eliminating $Q(s)$, we can simplify this to

$$\left.\begin{aligned}
sLI_1(s) + \frac{1}{sC}\left(I_1(s) - I_2(s)\right) &= V(s) \\[2ex]
sLI_2(s) + RI_2(s) + \frac{1}{sC}\left(I_2(s) - I_1(s)\right) &= 0
\end{aligned}\right\} \tag{3}$$

The great virtue of this method is that we can go straight from the circuit to the system of equations (2) without having to bother about differential equations at all. All that is necessary is to apply Kirchhoff's laws to the circuit, treating an inductor just as if it were a resistor with resistance sL, and a capacitor just as if it were a resistor with resistance $\frac{1}{sC}$. We can now solve the algebraic equations (3) for (say) I_2 in terms of V, obtaining

$$I_2(s) = \frac{sCV(s)}{(s^2LC + 1)(s^2LC + sRC + 1) - 1}$$

It is now possible to determine I_2, and hence its inverse transform i_2, for any $V = \mathcal{L}[v]$, that is for any input voltage function v. But this is not necessarily the way an electrical engineer would actually proceed. Electrical engineers usually base their design criteria on the case where the input (here v) is, say, a sine function, so that V is proportional to

$$\frac{a}{s^2 + a^2}$$

for some a. The Laplace transform I is then a rational function, and can be evaluated by the partial fraction methods we have described. (There are various tricks for speeding up the calculations, which could be rather tedious by the methods described in Section 29.2.)

Exercises

1. Without writing down any differential equations, obtain formulas giving $I(s)$, the Laplace transform of the current $i(t)$, in terms of R, C, and $V(s)$ the Laplace transform of the voltage $v(t)$ produced by the generator. Assume the initial current in the circuit and the initial charge on the capacitor to be zero.
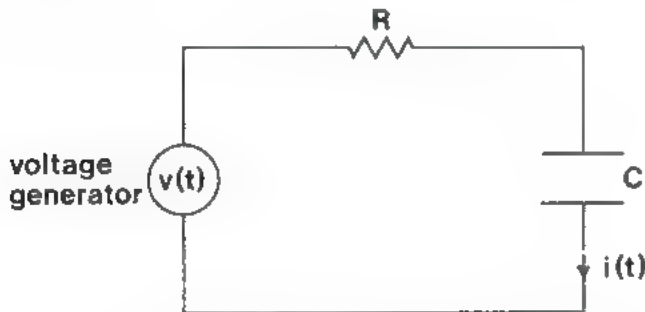
2. Without writing down any differential equations obtain $I_2(s)$ in terms of R_1, R_2, L, C, and $V(s)$. Assume the initial current in the network and the initial charge on the capacitor to be zero.
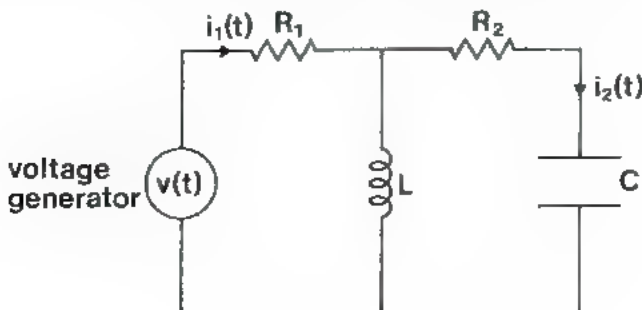
Solutions

1. Kirchhoff's second law applied to the Laplace transforms gives (see the discussion of Equation (2) above)

$$V(s) = RI(s) + \frac{1}{sC} I(s).$$

Thus

$$I(s) = \frac{V(s)}{R + \dfrac{1}{sC}}.$$

2. Kirchhoff's first law tells us that the downward current in the inductor is $i_1(t) - i_2(t)$. The second law then gives

$$V(s) = R_1 I_1(s) + sL(I_1(s) - I_2(s))$$

$$0 = -sL(I_1(s) - I_2(s)) + R_2 I_2(s) + \frac{1}{sC} I_2(s)$$

Solving for $I_2(s)$ in terms of $V(s)$, we obtain

$$I_2(s) = \frac{sLV(s)}{(sL + R_1)\left(sL + R_2 + \dfrac{1}{sC}\right) - s^2 L^2}.$$

29.3.3 Summary of Section 29.3

No new terms were defined in this section.

Technique

Solve a nonhomogeneous system of equations using Laplace transforms.

29.4 SUMMARY OF THE UNIT

Laplace transforms are related to a linear transformation $\mathcal{L}^*$ whose domain $\mathcal{E}^*$ and codomain $\mathcal{F}^*$ are vector spaces of equivalence classes of functions, defined as follows.

$\mathcal{E}^*$ consists of equivalence classes of piecewise continuous functions of exponential order, from $[0, \infty)$ to R, under the equivalence relation

$f \rho g$ if $f(t) = g(t)$ except at a finite number of points in every finite sub-interval of $[0, \infty)$

("Piecewise continuous" means that the function has at most a finite number of discontinuities in every finite sub-interval of $[0, \infty)$. "Of exponential order" means that there are constants C and a such that

$$|f(t)| \leqslant Ce^{at} \quad (t \geqslant 0).)$$

$\mathcal{F}^*$ consists of equivalence classes of continuous real functions with domains of the form (a, ∞). The equivalence relation here is:

$f \sim g$ if there is a constant c in the domain of both f and g, such that $f(s) = g(s)$ for all $s > c$.

The Laplace transform of f is defined by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) \, dt \qquad (f \in \mathcal{E})$$

and $\mathcal{L}^*$ is then defined to map the equivalence class containing f to the one containing $\mathcal{L}[f]$. The integral always converges for $s \in (c, \infty)$ for some real number c, whenever $f \in \mathcal{E}$.

Lerch's theorem states that $\mathcal{L}^*$ is one-to-one. Its image space is a proper subspace of $\mathcal{F}^*$, since

$$\lim_{s \to \infty} \mathcal{L}[f](s) = 0$$

for all $f \in \mathcal{E}$. The application of the Laplace transform to differential equations follows from the fact that, whenever $f \in \mathcal{E}$ and f has continuous derived functions up to the $(n-1)$th derived function and a piecewise continuous nth derived function, then

$$\mathcal{L}[D^n f](s) = s^n \mathcal{L}[f](s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) \quad (1)$$

This allows a differential equation in f to be converted into an algebraic equation in $\mathcal{L}[f]$. In particular, if we restrict f to have

$$f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0,$$

then for any polynomial P of degree n we have

$$\mathcal{L}[P(D)f](s) = P(s)\mathcal{L}[f](s)$$

giving a commutative diagram

Thus, to find a solution $y(t)$ to the differential equation

$$P(D)y = f,$$

with initial conditions

$$y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0,$$

we proceed as follows:

$$\mathcal{L}[P(D)y] - \mathcal{L}[f]$$

$$P(s)\mathcal{L}[y](s) - \mathcal{L}[f](s)$$

$$\mathcal{L}[y](s) = \frac{\mathcal{L}[f](s)}{P(s)}$$

$$y(t) - \mathcal{L}^{-1}\left[\frac{\mathcal{L}[f](s)}{P(s)}\right].$$

For other initial conditions, the more general Equation (1) can be employed.

The problem thus resolves itself to that of finding $\mathcal{L}^{-1}[f]$. We do this where possible by means of a table of certain standard Laplace transforms (page K229), and the two shifting theorems:

$$\mathcal{L}[e^{at}f(t)](s) = (\mathcal{L}[f])(s - a)$$

$$\mathcal{L}[u_a(t)g(t - a)](s) = e^{-as}\mathcal{L}[g](s) \qquad \text{(optional material)}$$

where u_a is the unit step function specified by

$$\begin{aligned} u_a(t) &= 0 \quad t \leqslant a\\ &= 1 \quad t > a \end{aligned}$$

In particular, using the first shifting theorem, the table of standard Laplace transforms, and the technique of partial fractions, it is possible to find the inverse Laplace transform of any rational polynomial function $\frac{p(s)}{q(s)}$ where degree of $p <$ degree of q.

Laplace transforms are especially useful in solving systems of linear differential equations and are used extensively in solving network problems in electrical engineering.

Definitions

Laplace transform	(page K179)
functions of exponential order	(page K180)
abscissa of convergence	(page K182)
unit step function	(page K194)
partial fraction expansion	(page C30)
rational function	(page C31)

Theorems

1. (5-1, page K181)
If f is a piecewise continuous function of exponential order, there exists a real number α such that

$$\int_0^\infty e^{-st}f(t)\, dt \qquad \cdot$$

converges for all values of $s > \alpha$.

2. (page C11)
If n is a non-negative integer and a is a positive real number, then

$$\lim_{t \to \infty} t^n e^{-at} = 0.$$

3. (5-2, page K185) (Lerch's Theorem)
Let f and g be piecewise continuous functions of exponential order, and suppose there exists a real number s_0 such that

$$\mathcal{L}[f](s) = \mathcal{L}[g](s)$$

for all $s > s_0$. Then, with the possible exception of points of discontinuity,

$$f(t) = g(t)$$

for all $t > 0$.

4. (5-3, page K185)

If f is a function of exponential order, then

$$\lim_{s \to \infty} \mathcal{L}[f] = 0.$$

5. (5-4, page K187)

Let f be continuous on $(0, \infty)$, and suppose that f' is piecewise continuous and of exponential order on $[0, \infty)$. Then

$$\mathcal{L}[f'] = s\mathcal{L}[f] - f(0^+)$$

where

$$f(0^+) = \lim_{t \to 0+} f(t).$$

More generally, if $f, f', \ldots, f^{(n-1)}$ are continuous for all $t > 0$, and if $f^{(n)}$ is piecewise continuous and of exponential order on $[0, \infty)$, then

$$\mathcal{L}[f''] = s^2\mathcal{L}[f] - sf(0^+) - f'(0^+)$$

$$\vdots$$

$$\mathcal{L}[f^{(n)}] = s^n\mathcal{L}[f] - s^{n-1}f(0^+) - s^{n-2}f'(0^+) - \cdots - f^{(n-1)}(0^+).$$

6. (5-6, page K193) (first shifting theorem)

If $\mathcal{L}[f] = \phi(s)$, then

$$\mathcal{L}[e^{at}f(t)] = \phi(s - a).$$

7. (page C29)

For a given polynomial function q the set V of all functions of the form

$$\frac{(\text{polynomial of degree} \leqslant n - 1)}{q}$$

is a vector space of dimension n, and if q is of degree n and has n distinct roots $a_1, \ldots, a_n$ then the set of n functions

$$\frac{1}{s - a_1}, \frac{1}{s - a_2}, \ldots, \frac{1}{s - a_n} \qquad (s \in R, s \neq a_1, \ldots, a_n)$$

is a basis for V.

8. (page C31)

If

$$q(s) = (s - a_1) \ldots (s - a_n)$$

with all the a_i distinct, and $p(s)$ is of degree $\leqslant n - 1$, then in

$$\frac{p(s)}{q(s)} = \sum_{i=1}^{n} \frac{\alpha_i}{(s - a_i)}$$

each α_i is given by

$$\alpha_i = \frac{p(a_i)}{\underbrace{(a_i - a_1)(a_i - a_2) \ldots (a_i - a_n)}_{\text{omitting } (a_i - a_i)}}.$$

Techniques

1. Find the abscissa of convergence of $\mathcal{L}[f]$ for suitable $f \in \mathcal{F}$.
2. Find $\mathcal{L}[f]$ for suitable $f \in \mathcal{F}$.
3. Given the Laplace transform of a function, find the function.
4. Use Laplace transforms to solve a constant-coefficient initial-value problem.
5. Given a rational function, find its partial fraction expansion.
6. Solve a nonhomogeneous system of equations using Laplace transforms.

Notation

$\mathcal{L}[f](s)$	(page K179)
$\mathcal{E}$	(page C13)
$\mathcal{F}$	(page C13)
$\mathcal{F}^*$	(page K184)
$\mathcal{E}^*$	(page C17)
$u_a(t)$	(page K194)

29.5 SELF-ASSESSMENT

Self-assessment Test

This Self-assessment Test is designed to help you test your understanding of the unit. It can also be used, together with the summary of the unit, for revision. The answers to these questions will be found on the next non-facing page. We suggest that you complete the whole test before looking at the solutions.

1. Which of the following functions are
   (a) piecewise continuous on their domains
   (b) of exponential order
   (c) in the domain of $\mathcal{L}$?

$$f : t \longmapsto \begin{cases} t^{21} & \text{if } t > 0 \\ 1.5 & \text{if } t = 0 \end{cases} \quad (t \in [0, \infty))$$

$$g : t \longmapsto \begin{cases} e^t & \text{if } t > \pi \\ \sin(t^{-1}) & \text{if } 0 < t < \pi \end{cases} \quad (t \in [0, \infty))$$

$$h : t \longmapsto \begin{cases} e^{t^2} & \text{if } t > 8 \\ \sin t & \text{if } 6 \leqslant t < 8 \end{cases} \quad (t \in [6, \infty))$$

2. Give the Laplace transform of

$$t \longmapsto e^t + \sin t + e^{-t} \cos 2t \quad (t \in [0, \infty)).$$

3. Express the following as partial fractions

   (i) $\dfrac{1}{s^2(s+1)}$

   (ii) $\dfrac{2s+3}{(s+2)(s^2-4s+1)}$

   (iii) $\dfrac{6+s-s^2+7s^3-s^4}{s^3(s-3)(s+2)}$

4. Give the inverse Laplace transform of

   (i) $s \longmapsto \dfrac{1}{s^2(s+1)} \quad (s \in [0, \infty))$

   (ii) $s \longmapsto \dfrac{e^{-s}}{s} \quad (s \in (0, \infty))$ (optional part of unit)

5. If

$$y''(t) + y(t) = \sin t,$$

   with $y(0) = 3$ and $y'(0) = 4$, find $\mathcal{L}[y](s)$ and hence the solution of the equation.

6. Solve the following initial-value problem using Laplace transforms

$$y''(t) - y'(t) - 6y(t) = 3t^2 + t - 1$$

   with $y(0) = -1, y'(0) = 6$.

7. Given the system

$$\left. \begin{array}{l} y_1'(t) + y_2(t) = 3t \\ y_2'(t) + 4y_1(t) = e^{-t} \end{array} \right\} \text{with } y_1(0) = 2, y_2(0) = 1$$

   find the expressions for the Laplace transforms of y_1 and y_2.

8. If the current in the circuit shown is $i(t)$ and the charge on the capacitor at time 0 is 0, write down (without writing down any differential equations) $\mathcal{L}[i](s)$ in terms of s and the numbers R, C and V.
(This question is based on an optional part of the unit.)
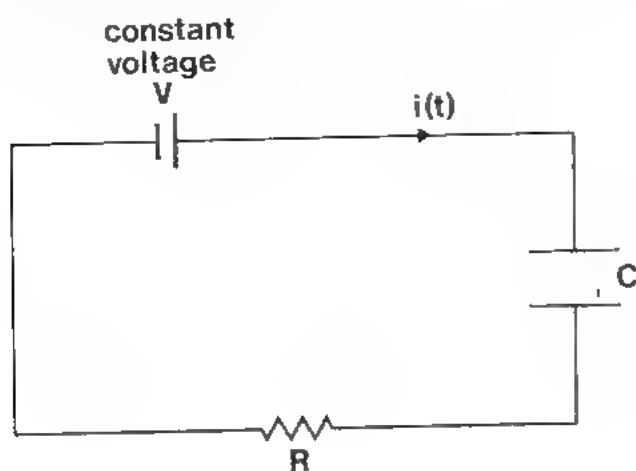
Solutions to Self-assessment Test

1. f: (a) Yes (b) Yes (c) Yes
   g: (a) No   (b) Yes (c) No
   h: (a) Yes (b) No   (c) No

2. $s \longmapsto \dfrac{1}{s-1} + \dfrac{1}{s^2+1} + \dfrac{s+1}{(s+1)^2+4}$

3. (i) $\dfrac{1}{s^2(s+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+1}$

$$= \frac{As(s+1) + B(s+1) + Cs^2}{s^2(s+1)}$$

This gives the equations

$$A + C = 0$$
$$A + B = 0$$
$$B = 1$$

hence

$$A = -1, B = 1, C = 1$$

and so

$$\frac{1}{s^2(s+1)} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$$

(ii) $\dfrac{2s+3}{(s+2)(s^2-4s+1)} = \dfrac{A}{s+2} + \dfrac{B}{(s^2-4s+1)} + \dfrac{Cs}{(s^2-4s+1)}$

$$= \frac{A(s^2-4s+1) + B(s+2) + Cs(s+2)}{(s+2)(s^2-4s+1)}$$

This gives the equations

$$A \qquad + C = 0$$
$$-4A + B + 2C = 2$$
$$A + 2B \qquad = 3$$

hence

$$A = -\tfrac{1}{13}, B = \tfrac{20}{13}, C = \tfrac{1}{13}$$

and so

$$\frac{2s+3}{(s+2)(s^2-4s+1)} = \frac{-1}{13(s+2)} + \frac{20}{13(s^2-4s+1)}$$
$$+ \frac{s}{13(s^2-4s+1)}$$

(iii) $\dfrac{6+s-s^2+7s^3-s^4}{s^3(s-3)(s+2)} = \dfrac{A}{s^3} + \dfrac{B}{s^2} + \dfrac{C}{s}$

$$+ \frac{D}{s-3} + \frac{E}{s+2}$$

$$= \frac{A(s-3)(s+2) + Bs(s-3)(s+2) + Cs^2(s-3)(s+2) + Ds^3(s+2) + Es^3(s-3)}{s^3(s-3)(s+2)}$$

This gives the equations

$$-6A = 6$$
$$-A - 6B = 1$$
$$A - B - 6C = -1$$
$$B - C + 2D - 3E = 7$$
$$C + D + E = -1$$

hence

$$A = -1, B = 0, C = 0, D = \tfrac{4}{5}, E = -\tfrac{9}{5}$$

and so

$$\frac{6 + s - s^2 + 7s^3 - s^4}{s^3(s-3)(s+2)} = \frac{-1}{s^3} + \frac{4}{5(s-3)} - \frac{9}{5(s+2)}$$

4. (i) $\dfrac{1}{s^2(s+1)} = -\dfrac{1}{s} + \dfrac{1}{s^2} + \dfrac{1}{s+1}$

(from Solution 3(i))

Hence

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)}\right] = -1 + t + e^{-t}$$

(ii) $\mathcal{L}^{-1}\left[\dfrac{e^{-s}}{s}\right] = u_1(t)$

5. $$s^2\mathcal{L}[y](s) - sy(0) - y'(0) + \mathcal{L}[y](s) = \frac{1}{s^2+1}$$

$$s^2\mathcal{L}[y](s) - 3s - 4 + \mathcal{L}[y](s) = \frac{1}{s^2+1}$$

$$\mathcal{L}[y](s) = \frac{1}{s^2+1}\left(\frac{1}{s^2+1} + 3s + 4\right)$$

hence

$$y(t) = \tfrac{1}{2}(\sin t - t \cos t) + 3 \cos t + 4 \sin t$$
$$= \tfrac{9}{2}\sin t + (3 - \tfrac{1}{2}t)\cos t$$

6. $$s^2\mathcal{L}[y](s) - sy(0) - y'(0) - s\mathcal{L}[y](s) + y(0) - 6\mathcal{L}[y](s)$$

$$= \frac{6}{s^3} + \frac{1}{s^2} - \frac{1}{s}$$

$$s^2\mathcal{L}[y](s) + s - 6 - s\mathcal{L}[y](s) - 1 - 6\mathcal{L}[y](s) = \frac{6}{s^3} + \frac{1}{s^2} - \frac{1}{s}$$

$$\mathcal{L}[y](s) = \frac{6 + s - s^2 + 7s^3 - s^4}{s^3(s-3)(s+2)}$$

$$\mathcal{L}[y](s) = -\frac{1}{s^3} + \frac{4}{5(s-3)} - \frac{9}{5(s+2)}$$

(from Solution 3(iii))

Hence $y(t) = -\tfrac{1}{2}t^2 + \tfrac{4}{5}e^{3t} - \tfrac{9}{5}e^{-2t}$

7. $$s\mathcal{L}[y_1](s) - 2 + \mathcal{L}[y_2](s) = \frac{3}{s^2}$$

$$s\mathcal{L}[y_2](s) - 1 + 4\mathcal{L}[y_1](s) - \frac{1}{s+1}$$

Solving for $\mathfrak{L}[y_1](s)$ and $\mathfrak{L}[y_2](s)$ gives

$$\mathfrak{L}[y_1](s) = \frac{1}{(s^2 - 4)} \left[\frac{3}{s} - \frac{1}{s+1} + 2s - 1 \right]$$

and

$$\mathfrak{L}[y_2](s) = \frac{3}{s^2} + 2 - \frac{s}{(s^2 - 4)} \left[\frac{3}{s} - \frac{1}{s+1} + 2s - 1 \right]$$

8. $\mathfrak{L}[i](s) = \dfrac{V}{R + \dfrac{1}{sC}}$

29.6 APPENDICES (Optional)

These appendices are notes to parts of Chapter 5 of K, which deal with aspects of the Laplace transform that are not part of our syllabus, but which you may find interesting. In particular, Theorem 5-11, page K214, gives a characterization of the Green's function for a constant-coefficient linear differential operator.

Appendix 1   The Convolution Theorem

Notes to Section 5-7, pages K206–209.

(i)   *Equation (5-33), page* K206   The integrand here is the function

$$\xi \longmapsto f(t - \xi)g(\xi) \qquad (\xi \in [0, t])$$

Different values of t give different values for the definite integral; thus varying t produces a function, and it is this function whose Laplace transform is

$$s \longmapsto \phi(s)\psi(s).$$

(ii)   *Equation (5-35), page* K206   The function described in note (i) is here given a name and a notation. $\mathcal{L}$ is a morphism from $\mathcal{E}$ with binary operation $*$ to $\mathcal{F}$ with the binary operation of multiplication of functions

$$(f, g) \xrightarrow{\qquad * \qquad} f * g$$

$$\mathcal{L} \downarrow \qquad\qquad\qquad\qquad \downarrow \mathcal{L}$$

$$(\mathcal{L}[f], \mathcal{L}[g]) \xrightarrow{\quad \text{multiply} \quad} \mathcal{L}[f]\mathcal{L}[g] = \mathcal{L}[f * g]$$

The convolution product has applications in algebra as well as analysis.

(iii)   *line 4, page* K207   Multiple integration is not part of this course, but it is analogous to integration in one variable: instead of integrating over intervals of $\mathcal{R}$, one integrates over regions of $\mathcal{R}^2$. For any particular region in $\mathcal{R}^2$, there are generally several ways of describing how to restrict the pair (ξ, t) to lie within the region. In particular, one can restrict t first, then describe the restriction on ξ, in terms of t; or vice versa.

Appendix 2   Green's Function and the Laplace Transform

Notes to Section 5-8, pages K212–216.

(i)   *Equation (5-47), page* K214   Using the formula

$$\mathcal{L}[D^k y](s) = s^k \mathcal{L}[y](s) - s^{k-1} y(0) - \cdots - y^{(k-1)}(0)$$

for $k = 1, 2, \ldots, n$, we see that, if y obeys the conditions in (5-47), then

$$\mathcal{L}[p(D)y](s) = p(s)\mathcal{L}[y](s) - 1$$

since the leading coefficient of the polynomial p is 1 by Equation (5-43).
Thus the equation

$$Ly = 0$$

(i.e. in our notation, $p(D)y = 0$), becomes, when the Laplace transform is applied to it,

$$p(s)\mathcal{L}[y](s) - 1 = 0$$

i.e.

$$\mathcal{L}[y](s) = \frac{1}{p(s)}$$

and so

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t).$$

(ii)   *line 3, Theorem 5-11, page* K214   The "auxiliary polynomial" of the operator $p(D)$, is the polynomial p.

Appendix 3   The Vibrating Spring: Impulse Functions

Notes to Section 5-9, pages K218–225.

The first part of this section on the vibrating spring is very straightforward. The notes are all concerned with the concept of an *impulse function*.

(i)   *footnote, page* K222   More generally, a force $F(t)$ acting in a given direction (but dependent on time) imparts an impulse

$$\int_{t_0}^{t_1} F(t)\, dt$$

to the body between times t_0 and t_1.

This impulse is again equal to the change of momentum of the body due to the influence of the force over this period.

When we give a sharp blow to an object, we succeed in imparting momentum to it. In other words,

$$\int_{t_0}^{t_1} F(t)\, dt = \int_{a}^{a+\tau} F(t)\, dt \neq 0$$

where the blow occurred between a and $a + \tau$. But the interval $[a, a + \tau]$ can be narrowed down to be as small as we please, as long as it contains the instant a at which the blow took place. As we saw in sub-section 12.3.2 of *Unit 12, Linear Functionals and Duality*, this implies that F cannot be a function as ordinarily defined (though it can be given an adequate meaning, as a linear functional).

(ii)   *Equations (5-63), page* K223, *and (5-66), page* K224   In the notation of *Unit 12*, sub-section 12.3.2, $\delta(t)$ and $\delta(t - a)$ are $\delta_0(t)$ and $\delta_a(t)$ respectively. A lot of the working can be short-circuited at this point, by noting that in sub-section 12.3.2 we *defined* $\delta_a(t)$ in such a way that

$$\int_{b}^{c} f(t)\, \delta_a(t)\, dt = f(a)$$

whenever $a \in [b, c]$. Thus, by definition,

$$\mathcal{L}[\delta(t - a)] = \mathcal{L}[\delta_a(t)] = \int_{0}^{\infty} e^{-st} \delta_a(t)\, dt$$
$$= e^{-as}.$$

LINEAR MATHEMATICS